Math 32404: Advanced Calculus II: Final Exam Review

Material Covered: Munkres §1-8 and 12-22. In Chapter 1, I will try to focus questions on §1, 3, 7, and 8.

Definitions. You will be asked to define several terms on the test. **These terms normally have one definition, as given in the book. You are expected to know this definition. Exceptions are in the comments below.** The following is a list of terms which might appear.

power set, arbitrary union, arbitrary intersection, relation, equivalence relation, partition, order relation, immediate successor/predecessor, dictionary order, largest/smallest element, upper and lower bounds, greatest lower bound (infimum), least upper bound (supremum), greatest lower bound property, least upper bound property, finite, infinite, countably infinite, countable, uncountable, recursion formula, have the same cardinality, topology, open set, discrete topology, trivial topology, finer, strictly finer, coarser, strictly coarser, comparable (topologies), basis for a topology, topology generated by a basis, standard topology on $\mathbb{R}$*, lower limit topology (*$\mathbb{R}_\ell$*), subbasis, topology generated by a subbasis, order topology, interval, open interval, closed interval, half-open interval, rays, product topology (both for finite and infinite products), projection, standard topology on* $\mathbb{R}^n$*, subspace topology, open in a subset, convex subset of an ordered set, closed set, closed in a subset, interior, closure, intersecting sets, neighborhood, limit point, convergence (of a sequence), limit of a sequence, Hausdorff space,* T_1*-axiom, boundary of a subset, continuous function (between topological spaces), continuous at a point, homeomorphism, topological property, imbedding, coordinate functions, tuple, coordinates in an arbitrary product, cartesian product, box topology, product space, metric, triangle inequality,* ϵ*-ball centered at a point, metric topology, metric space, metrizable, bounded set, standard bounded metric, euclidean metric, square or max metric, uniform metric, uniform topology, basis at a point (or neighborhood basis), first countability axiom, uniform convergence, isometric imbedding*

Results you should know. Below is a list of results you should know and be able to apply. Results I think you should be able to prove are in bold.

Equivalence classes form a partition, The least upper bound property holds if and only if the greatest lower bound property holds, Criteria for countability (Thm 7.1), A subsets of a countable sets is countable, A countable union of countable sets is countable, A finite product of countable sets is countable, The set $\{0,1\}^\omega$ *is uncountable,* **There is no surjective map** $A \to \mathcal{P}(A)$ *(and no injective map* $\mathcal{P}(A) \to A$*), The Schroeder-Bernstein Theorem (Exercise §7 # 6), The principle of recursive definition (Theorem 8.4),* **Characterization of sets in a topology generated by a basis (Thm 13.1)***, Criterion for one topology to be finer than another in terms of bases (Lemma 13.3),* **Basis for the product topology in** $X \times Y$ **(Thm 15.1)***, Basis for the subspace topology (Lemma 16.1), If* $A \times B \subset X \times Y$*, then the subspace and product topologies on* $A \times B$ *are* *the same (Thm 16.3), For convex subsets of an ordered space, the order and subspace topologies coincide (Thm 16.4),* **Set theoretic properties of the collection of closed sets (Thm 17.1)**, *Characterization of closed sets in the subspace topology (Thm 17.2), If $A \subset B$ is a closed subset in B and $B \subset C$ is closed in C, then $A \subset C$ is closed in C (Thm 17.3), Interaction between closure and the subspace topology (Thm 17.4),* **Neighborhood characterization of closure (Thm 17.5)**, *Limit point characterization of closure (Thm 17.6),* **Finite subsets of Hausdorff spaces are closed (Thm 17.8)**, *Characterization of limit points in T_1 spaces (Thm 17.9),* **Limit points in Hausdorff spaces are unique (Thm 17.10)**, *Many spaces have the Hausdorff topology (Thm 17.11), Characterizations of continuity (Thm 18.1), Rules for constructing continuous functions (Thm 18.2), The pasting lemma (Thm 18.3),* **Maps into products (Thms 18.4, 19.6)**, *Comparison of the box and product topologies (Thm 19.1), Bases of the box and product topologies (Thm 19.2), Interaction of product spaces and subspaces (Thm 19.3),* **Hausdorff product is preserved under arbirary products (Thm 19.4)**, *Closures of products of subsets (Thm 19.5), The standard bounded metric assoicated to d induces the same topology as d (Thm 20.1), Comparing metric topologies (Thm 20.2), The topologies on $\mathbb{R}^n$ induced by the Euclidean and square topologies are the same, Comparison of box, product and uniform topologies on a product of metric spaces (Thm 20.4), A countable product of metric spaces is metrizable (Thm 20.5),* **Characterization of continuity for maps between metric spaces (Thm 21.1)**, *The sequence lemma (Lem 21.2), Continuity and convergent sequences (Thm 21.3), Algebraic operations on $\mathbb{R}$ are continuous (Lem 21.4), Algebraic operations on functions into $\mathbb{R}$ preserve continuity (Thm 21.5), The uniform limit theorem (Thm 21.6)*

Example questions: Remark. I wouldn't take too much stock in these questions. The homework problems assigned and other problems in the book are probably better as these were hastily written and not used on a previous test. Please also take a look at the "additional problems" assigned but not collected for homework.

1. (a) Complete the following definition:
   A topological space X is called a *Hausdorff space* if ...

   (b) Prove that any finite subset of a Hausdorff space is closed.

2. (a) Complete the following definition: A *homeomorphism* between two topological spaces, $f : X \to Y$, is ...

   (b) Two topological spaces X and Y are said to be *homeomorphic* if there is a homeomorphism $X \to Y$. We denote Let $\{X_\alpha : \alpha \in I\}$ be a collection of topological spaces. Write $X_\alpha \simeq X_\beta$ if X_α and X_β are homeomorphic. Prove that $\simeq$ is an equivalence relation on $\{X_\alpha\}$.

3. Let X be an uncountable set and let

$$\mathcal{T} = \{A \subset X : A = \varnothing \text{ or } X \setminus A \text{ is countable}\}.$$

   (a) Prove that $\mathcal{T}$ is a topology on X. (It is called the *cocountable topology.*)

(b) Prove that this topology is not metrizable.

4. Consider the following collection of subsets of $\mathbb{R}$:

$$\mathcal{C} = \{(a,b) \cup (a+1, b+1) : \ a, b \in \mathbb{R} \text{ and } a < b\}.$$

   (a) Prove that $\mathcal{C}$ is not a basis for a topology on $\mathbb{R}$.

   (b) Prove that $\mathcal{C}$ is a subbasis for a topology on $\mathbb{R}$.

   (c) Prove that this subbasis generates the ordinary topology on $\mathbb{R}$.

5. (a) Let X and Y be topological spaces and $A \subset X$ and $B \subset Y$ be closed sets. Prove that $A \times B$ is a closed subset of $X \times Y$. (This is Munkres §17 # 3.)

   (b) Let $\{X_\alpha : \ \alpha \in J\}$ be an arbitrary collection of topological spaces. Suppose for each $\alpha \in J$, $A_\alpha \subset X_\alpha$ is a closed subset. Let $X = \prod_{\alpha \in J} X_\alpha$ and $A = \prod_{\alpha \in J} A_\alpha$. Then $A \subset X$. Is A necessarily closed as a subset of X? If so, prove it. If not, give a counterexample.