Topology: Midterm Review 2

Material Covered: Munkres §9-10, §22-24, and §26-28. The course is cumulative, so it is also important to understand material from earlier in the course.

Things to study: The textbook, your class notes, the homework problems and solutions (including problems mentioned but not collected), the problems included here.

Definitions:

You will be asked to define several terms on the test. **These terms normally have one definition, as given in the book. You are expected to know this definition.** The following is a list of terms which might appear.

Axiom of choice, choice function, finite axiom of choice, well-ordered, section, minimal uncountable well-ordered set, quotient map, saturated, open map, closed map, quotient topology, quotient space, separation, connected, totally disconnected, linear continuum, path, path connected, cover, covering, open covering, compact, cover of a subset, finite intersection property, nested sequence, distance between a point and a subset of a metric space, Lebesgue number, isolated point, limit point compact, subsequence, sequentially compact,

Results you should know:

Below is a list of results you should know and be able to apply. Results I think you should be able to prove are in bold.

Axiom of choice, Existence of a choice function, Finite ordered sets are well-ordered (Thm 10.1), Well-ordering theorem, There is a smallest uncountable well-ordered set (Lemma 10.2), **Upper bounded property of the smallest uncountable well-ordered set (Thm 10.3)**, *Subspaces and quotients (Thm 22.1), Constructing continuous functions from a quotient space (Thm 22.2, Cor 22.3), Connectedness and subspaces (Lemma 23.1),* **Connected subsets of a space with a separation (Lemma 23.2)**, **The union of connected sets with a common point is connected (Thm 23.3)**, *A subspace formed from a connected space by adding limit points is still connected,* **The image of a connected space under a continuous map is continuous (Thm 23.3)**, *Finite products of connected spaces are connected (Thm 23.6), Linear continua are connected (Thm 24.1), The real line is connected (Cor 24.2),* **Generalized Intermediate Value Theorem (Thm 24.3)**, **Path connected spaces are connected (Page 155)**, *Compactness of subspaces (Lemma 26.1),* **Closed subsets of compact spaces are compact (Thm 26.2)**, **Compact subspaces of a Hausdorff space are closed (Thm 26.3)**, **Compact subsets of a Hausdorff space can be separated from points (Lemma 26.4)**, **Continuous images of compact sets are compact (Thm 26.5)**, *Continuous bijections are homeomorphisms when the codomain is Hausdorff (Thm 26.6), Finite products of compact spaces are compact (Thm 26.7), Characterization of compactness in terms of the finite intersection property (Thm 26.9), Intersections of a nested sequence* *of non-empty compact sets are non-empty, Closed intervals in $\mathbb{R}$ and similar ordered spaces are compact (Thm 27.1, Cor 27.2), Characterization of compact subsets of $\mathbb{R}^n$ (Thm 27.3),* **Generalized extreme value theorem (Thm 27.4)**, *The Lebesgue number lemma (Lemma 27.5),* **A non-empty compact Hausdorff space without isolated points is uncountable (Thm 27.7)**, *Compactness implies limit point compactness (Thm 28.1), Equivalence of forms of compactness for a metrizable space (Thm 28.2),*

Example questions:

1. Suppose $(A, <_A)$ and $(B, <_B)$ are disjoint well-ordered sets. Let $C = A \cup B$ and define the ordering on C by $c_1 <_C c_2$ for $c_1, c_2 \in C$ if any of the following statements are true:

   - $c_1, c_2 \in A$ and $c_1 <_A c_2$.
   - $c_1, c_2 \in B$ and $c_1 <_B c_2$.
   - $c_1 \in A$ and $c_2 \in B$.

   Prove that $(C, <_C)$ is well-ordered.

2. Let $(X, <)$ be a well-ordered set with a maximal element x_+. Recall that the *section* by $\alpha \in X$ is
$$S_\alpha = \{x \in X : \ x < \alpha\}.$$
   Let A be a nonempty set. Suppose that there is no injective function $S_{x_+} \to A$. Prove that there is a unique $y \in X$ satisfying the two properties:

   - There is no injective function $S_y \to A$.
   - If $x < y$, then there is an injective function $S_x \to A$.

3. Let X be a Hausdorff space and $p : [a, b] \to X$ be a path in X. (That is, $[a, b] \subset \mathbb{R}$ is a closed interval, and p is continuous.) Let $Y = p([a, b])$ be the image, endowed with the subspace topology. Prove that the path p with restricted codomain, $\hat{p} : [a, b] \to Y$ is a quotient map.

4. Let X be a space and $\mathcal{T}$ and $\mathcal{T}'$ be topologies. Suppose $\mathcal{T}'$ is finer than $\mathcal{T}$, i.e., $\mathcal{T}' \supset \mathcal{T}$. Prove that if $(X, \mathcal{T}')$ is connected, then $(X, \mathcal{T})$ is connected.

5. Prove that if X is a connected space, and $f : X \to Y$ is continuous, then $f(X)$ is a connected subspace of Y.

6. (a) Recall that a path in a space X is a continuous map $f : [a, b] \to X$, where $[a, b] \subset \mathbb{R}$ is a closed interval. Complete following definition:
   A space X is *path connected* if ...

(b) Let X be a space. Suppose $a < b < c$ are real numbers. Suppose $f : [a, b] \to X$ and $g : [b, c] \to X$ are paths such that $f(b) = g(b)$. Prove that the function $h : [a, c] \to X$ defined by

$$h(x) = \begin{cases} f(x) & \text{if } x \in [a, b], \\ g(x) & \text{if } x \in [b, c] \end{cases}$$

is also a path.

(c) Prove that the product of two path connected spaces is path connected.

7. Let X be a Hausdorff space and $Y \subset X$ be a compact subspace. Suppose that $x_0 \in X \setminus Y$. Prove that there exists disjoint open sets $U, V \subset X$ such that $x_0 \in U$ and $Y \subset V$.

8. (Munkres §26 # 7) Show that if Y is compact, then the projection $\pi_1 : X \times Y \to X$ is a closed map.

9. (Munkres §26 # 8) Prove the following Theorem: Let $f : X \to Y$; let Y be compact Hausdorff. Then f is continuous if and only if the **graph** of f,

$$G_f = \{(x, f(x)) \mid x \in X\},$$

is closed in $X \times Y$. [Hint: If G_f is closed and V is a neighborhood of $f(x_0)$, then the intersection of G_f and $X \times (Y - V)$ is closed. Apply Exercise 7.]

10. Recall that a *neighborhood basis* for a point x in a space X is a collection $\mathcal{N}$ of neighborhoods of x with the property that if U is any neighborhood of x, then there is an $N \in \mathcal{N}$ such that $N \subset U$. A space X is *first countable* if every point in X has a countable neighborhood basis.

    Suppose X is a first countable space satisfying the T_1 axiom. Prove that if X is limit point compact, then X is sequentially compact.