Topology: Final Exam Review

Exam makeup:

- One problem covering definitions from algebraic topology, §51-55 and §58.
- Two to three additional problems covering algebraic topology, §51-55 and §58.
- Two to three problems covering earlier material divided between material from the first midterm (§1-8 and §12-22) and the second midterm (§9-10, §22-24 and §26-28). (Suggestion: Review the proofs of bolded results from previous review sheets.)

Things to study: The textbook, your class notes, the homework problems and solutions (including problems mentioned but not collected), the problems included here.

Definitions:

You will be asked to define several terms on the test. **These terms normally have one definition, as given in the book. You are expected to know this definition.** The following is a list of terms which might appear (all from §51-55 and §58.)

homotopic, homotopy, nulhomotopic (or null homotopic), path, initial point, final point, path homotopic, path homotopy, product of paths ($f * g$)*, reverse of a path* ($\bar{f}$ *if* f *is a path*)*, contractible, (group) homomorphism, kernel, isomorphism, loop, fundamental group (or first homotopy group), base point, simply connected, homomorphism induced by a continuous map, evenly covered, slices, covering map, covering space, local homeomorphism, figure-eight space, lifting of a map (in the presence of a covering map), lifting correspondence, retraction, retract, deformation retract, deformation retraction, homotopy equivalence, homotopy inverse, homotopy type,*

Results you should know:

Below is a list of results you should know and be able to apply. Results I think you should be able to prove are in bold.

Homotopic and path homotopic are equivalence relations (Lemma 51.1), **Properties of the concatenation of paths operation up to homotopy (Thm 51.2)***, Splitting up paths (Thm 51.3), Change of basepoint by conjugating paths by concatenation induces a homomorphism between fundamental groups (Thm 52.1), Varying basepoints yields isomorphic fundamental groups (Cor 52.2), Functorial properties of induced maps (Thm 52.4),* $\mathbb{R}$ *covers* S^1 *(Thm 53.1),* **Restricting covering maps (Thm 53.2),** *The product of covering maps is a covering map (Thm 53.3), Uniqueness of lifts of paths (Lemma 54.1), Uniqueness of lifts of path homotopies (Lemma 54.2), Lifts of homotopic paths have the same endpoints (Thm 54.3), Properties of the lifting correspondence (Thm 54.4),* $\pi_1(S^1)$ *is isomorphic to* $\mathbb{Z}$*,* **Inclusion of a retract into the ambient space induces an injective homomorphism between the fundamental groups (Lemma 55.1)***,* **There is no**

retraction of the 2-dimensional ball onto its boundary (Thm 55.2), *Equivalences to a map $S^1 \to X$ being null homotopic (Thm 55.3), A map from the closed 2-ball to itself always has a fixed point (Thm 55.6), The inclusion of S^n into $\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$ is an isomorphism of fundamental groups (Thm 58.2),* **The inclusion of a deformation retract into its ambient space induces an isomorphism of fundamental groups (Thm 58.3)**, *Homotopy equivalence induces an isomorphism of fundamental groups (Thm 58.7),*

Example questions:

1. Let $X = \{(x, y) \in \mathbb{R}^2 : 1 \leq |x| + |y| \text{ and } \max(|x|, |y|) \leq 1\}$. Let $\mathbf{p} \in X$. Describe the fundamental group $\pi_1(X, \mathbf{p})$ up to isomophism. Prove your answer is correct. You may use any of the results mentioned above.

2. Let X be a space and $x_0, x_1 \in X$. Let α be a path from x_1 to x_0 and β be a path from x_0 to x_1. Consider the map
$$\Phi : \pi_1(X, x_0) \to \pi_1(X, x_1); \quad [f] \mapsto [\alpha * f * \beta].$$
Consider the statement: "Φ is a group homomorphism." Prove that this statement is sometimes false by constructing a counterexample.

3. Let α be a path in X from x_0 to x_1; let β be a path in X from x_1 to x_2. Show that if $\gamma = \alpha * \beta$, then $\hat{\gamma} = \hat{\beta} \circ \hat{\alpha}$. (This is Munkres' §52 #2)

4. Recall that the map $p : \mathbb{R} \to S^1$ defined by
$$p(t) = \big(\cos(2\pi t), \sin(2\pi t)\big)$$
is a covering map.

   Let $a < b$ be real numbers. Prove that the restriction
$$p|_{(a,b)} : (a, b) \to S^1$$
is not a covering map.

5. Consider the maps $g, h : S^1 \to S^1$ given by $g(z) = z^n$ and $h(z) = 1/z^n$. (Here we represent S^1 as the set of complex numbers z of absolute value 1.) Compute the induced homomorphisms g_*, h_* of the infinite cyclic group $\pi_1(S^1, b_0)$ into itself. [Hint: Recall the equation $(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta$.] (This is Munkres §54 # 6.)

6. Let $T = S^1 \times S^1$ be the 2-dimensional torus. Let $p \in T$ and let $X = T \setminus \{p\}$. Find a deformation retraction onto a subset homeomorphic to a figure-eight. You may use pictures to describe your deformation retraction. (See Munkres' §58 Example 2 for an example of such pictures. This is easiest to see if you think of the torus as the square with opposite sides glued together by translation and take the point p to be the center of the square.)