Math 32404: Advanced Calculus II: Midterm 1 Review

Material Covered: §8.2-8.5

Definitions. You will be asked to define several terms on the test. **These terms all have one definition, as given in the book. You are expected to know this definition.** The following is a list of terms which might appear. (Others might appear as well).

norm on a vector space, euclidean norm, operator norm, matrix, critical point, Lipschitz function (Example 7.5.13), change of basis, sup or max norm (Exercise 8.2.3), L^1 norm (Exercise 8.2.4), eigenvalue and eigenvector, differentiable, derivative, partial derivative, gradient, curve, tangent vector, Jacobian determinant, continuously differentiable or C^1, locally invertible

Comments

- The book defined eigenvalue on page 33, but didn't seem to define eigenvector. Supposing A is an $n \times n$ matrix, a complex number $\lambda \in C$ is an *eigenvector* if there is a non-zero vector $x \in \mathbb{C}^n$ such that $Ax = \lambda x$. Any non-zero vector x such that there is a $\lambda \in \mathbb{C}$ satisfying $Ax = \lambda x$ is called an *eigenvector*.
- The book uses the term "locally invertible" but didn't formally define it. A function between metric spaces $f : X \to Y$ is *locally invertible* at $x \in X$ if there is an open set $U \subset X$ such that $x \in U$, $f(U)$ is open, and $f|_U : U \to f(U)$ is a bijection. Local invertibility is a consequence of the Inverse function theorem.
- The book didn't seem to formally define critical point. A *critical point* of a differentiable function f is an x in the domain such that $Df(x)$ is zero.

Results you should know. Below is a list of results you should know, and be able to apply. Results I think you should be able to prove are in bold. You will be asked to prove at least one of these results on the midterm.

Cauchy-Schwarz inequality (Lemma 7.1.4 and Thm 8.2.2), **Properties of the operator norm in finite dimensional spaces (Props 8.2.4-5)**, *The map $A \mapsto A^{-1}$ is continuous (Prop 8.2.6), Properties of the determinant (Props 8.2.8-10), Uniqueness of the derivative (Prop 8.3.2),* **Differentiable implies continuous (Prop 8.3.5)**, *The derivative is a linear operator (Prop 8.3.6), Matrix representation of the derivative (Prop. 8.3.9), Bounded derivative on a convex domain implies Lipschitz (Prop 8.4.2), Polynomials are C^1, Inverse function theorem, Implicit function theorem*

Example questions: These are some problems I have written for exams in the past.

1. Are the functions below differentiable at $\mathbf{0}$? Justify your answer with a proof.

$$\mathbf{f}(x,y,z) = \sqrt[3]{xyz} \quad \text{and} \quad \mathbf{g}(x,y,z) = \sqrt[3]{x^2yz}.$$

2. Let $g(a, b, c)$ be a C^2 function $\mathbb{R}^3 \to \mathbb{R}$. Let $f(x, y) = g(x^2, xy, y^2)$.
   (a) Compute $\frac{\partial f}{\partial x}(x, y)$ in terms of x, y and partial derivatives of g.
   (b) Compute $\frac{\partial^2 f}{\partial y \partial x}(x, y)$ in terms of x, y and partial derivatives of g.

3. (a) Complete the following definition. Let $S \subset \mathbb{R}^n$ be an open set. The function $f : S \to \mathbb{R}$ is differentiable at $\mathbf{a} \in S$ if
   (b) Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies $f(0, 0) = 0$ and

$$x - x^2 - y^2 \le f(x, y) \le x + x^2 + y^2 \quad \text{for all } (x, y) \in \mathbb{R}^2.$$

   Prove that f is differentiable at $(0, 0)$.

4. Show that the two side-lengths of an isosceles triangle can be expressed as a function of the perimeter and area of the triangle locally near any non-equilateral isosceles triangle.

5. Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a differentiable function. Define $g(x, y) = |f(x, y)|$ where $|\cdot|$ denotes the Euclidean norm.
   (a) Give a formula for $\nabla g(x, y)$ at any point (x, y) for which $f(x, y) \ne (0, 0)$. Your formula should be in terms of x, y, the entries of $f(x, y)$, and the partial derivatives of the two component functions of f.
   (b) Show that if $Df(x, y)$ is the zero matrix, then (x, y) is a critical point of g. That is, show that $Df(x, y) = 0$ implies that $Dg(x, y) = 0$.

6. Let $\alpha : \mathbb{R} \to \mathbb{R}^3$ be a parameterized C^1 path in the plane $z = 0$ and let $\beta : \mathbb{R} \to \mathbb{R}^3$ be a parameterized C^1 path in the plane $z = 1$. Define

$$F : \mathbb{R}^3 \to \mathbb{R}^3 \quad \text{by} \quad F(s, t, z) = \alpha(s) + z\big(\beta(t) - \alpha(s)\big).$$

   Then, when s and t are held constant, the map $z \mapsto F(s, t, z)$ parameterizes the line joining $\alpha(s)$ to $\beta(t)$.
   (a) Use the Inverse Function Theorem to show that if $z_0 \notin \{0, 1\}$ and $\alpha'(s_0)$ and $\beta'(t_0)$ are not parallel, then F is invertible in a neighborhood of the point (s_0, t_0, z_0) in the domain.

7. Let $\mathbf{G}(u, v) = (uv - 3, u^2 + 2v)$.
   (a) Suppose V is an open neighborhood of $(-3, 6) \in \mathbb{R}^2$ and that $\mathbf{H} : V \to \mathbb{R}^2$ is a map so that $\mathbf{G} \circ \mathbf{H}(x, y) = (x, y)$ for all $(x, y) \in V$. ($\mathbf{H}$ is a local inverse to $\mathbf{G}$ at $(0, 3)$.) Find the matrix of partial derivatives $D\mathbf{H}(-3, 6)$.
   (b) Let B_ϵ denote the open ball of radius ϵ at $(0, 3)$. Estimate the quantity

$$\text{Area } \mathbf{G}(B_\epsilon)/\text{Area } B_\epsilon$$

   for ϵ small. (Concretely, what is the limit of this ratio as $\epsilon \to 0$?)