BOUNDING THE DERIVATIVE

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This is a slightly different proof of the Lebl's Proposition 8.4.2.

In the hopes of avoiding confusion, for $f : \mathbb{R}^n \to \mathbb{R}^m$, I will use $Df \in L(\mathbb{R}^n, \mathbb{R}^m)$ to denote the derivative of a linear map. I will use something like $h'(x) \in \mathbb{R}$ to denote the derivative of a function $h : \mathbb{R} \to \mathbb{R}$. As we discussed in class, the corresponding linear map $Dh(x)$ is the map $t \mapsto h'(x)t$.

To avoid confusion between the one-variable derivative of a function $h : \mathbb{R} \to \mathbb{R}$ and the derivative of a

Proposition 1. *Let $U \subset \mathbb{R}^n$ be open and convex, let $f : U \to \mathbb{R}^m$ be differentiable and assume that*

$$\|Df(p)\| \leq M \quad \textit{for all } p \in U.$$

Then f is M-Lipschitz:

$$\|f(q) - f(p)\| \leq M\|q - p\| \quad \textit{for all } p, q \in U.$$

Proof. Fix $p, q \in U$. Observe that if $f(p) = f(q)$, then the conclusion is true because $\|f(q) - f(p)\| = 0$. So we will assume for the remainder of the proof that $f(p) \neq f(q)$.

Let $\ell : \mathbb{R} \to \mathbb{R}^n$ be the function defined by $\ell(t) = (1 - t)p + tq$. Since ℓ is continuous and U is open and convex and contains p and q, there is an open interval I containing $[0, 1]$ such that $\ell(I) \subset U$. Observe that ℓ is affine linear, so

$$D\ell(t) : s \mapsto s(q - p) \quad \text{for every } s, t \in \mathbb{R}.$$

Then we also have that $\|D\ell(t)\| = \|q - p\|$ for every $t \in \mathbb{R}$; see Exercise 8.2.6.

Now let $u = \frac{1}{\|f(q)-f(p)\|}(f(q) - f(p))$ which is a well defined unit vector since $f(p) \neq f(q)$. Note that the dot product with u gives the length of the perpendicular projection of a vector onto a line parallel to u. We define

$$g : \mathbb{R}^m \to \mathbb{R} \quad \text{by} \quad g(v) = u \cdot \mathbf{v}.$$

The map g is linear, so $Dg(v) = g$ for every $v \in \mathbb{R}^m$. Observe that $\|Dg(v)\| = 1$ for every v; again see Exercise 8.2.6.

Finally, define $h : \mathbb{R} \to \mathbb{R}$ by $h = g \circ f \circ \ell$. Observe that $h(0) = u \cdot f(p)$ and $h(1) = u \cdot f(q)$, so

$$\begin{aligned} h(1) - h(0) = u \cdot \big(f(q) - f(p)\big) &= \frac{1}{\|f(q) - f(p)\|}(f(q) - f(p)) \cdot \big(f(q) - f(p)\big) \\ &= \frac{1}{\|f(q) - f(p)\|}\|f(q) - f(p)\|^2 = \|f(q) - f(p)\|. \end{aligned}$$

By the Mean Value Theorem, there is a $t \in (0, 1)$ such that

$$\|f(q) - f(p)\| = h(1) - h(0) = \frac{h(1) - h(0)}{1 - 0} = h'(t). \tag{1}$$

Date: March 20, 2024.

Since $Dh(t)$ is the linear map $s \mapsto h'(t)s$, we have $\|Dh(t)\| = |h'(t)|$. Applying the chain rule, we have

$$Dh(t) = Dg\big(f \circ \ell(t)\big) \circ Df\big(\ell(t)\big) \circ D\ell(t).$$

The operator norm of a composition of linear maps is less than or equal to the product of the operator norms (see Proposition 8.2.5), so

$$\|Dh(t)\| \leq \|Dg\big(f \circ \ell(t)\big)\| \cdot \|Df\big(\ell(t)\big)\| \cdot \|D\ell(t)\| \leq 1 \cdot M \cdot \|q - p\|. \tag{2}$$

Combining equations (1) and (2), we see that

$$\|f(q) - f(p)\| = |h'(t)| = \|Dh(t)\| \leq M \cdot \|q - p\|.$$

as desired. □

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