Advanced Calculus II: Midterm 2 Review

Material Covered: §8.1-8.6, 9.2

Definitions. You will be asked to define several terms on the test. **These terms all have one definition, as given in the book. You are expected to know this definition.** The following is a list of terms which might appear. (Others might appear as well).

ector space, vector, (vector) subspace, linear combination, span, linearly independent, basis, dimension, linear map, invertible linear map, convex subset of a vector space, convex combination, convex hull, norm on a vector space, euclidean norm, operator norm, matrix, critical point, Lipschitz function (Example 7.5.13), change of basis, sup or max norm (Exercise 8.2.3), L^1 norm (Exercise 8.2.4), eigenvalue and eigenvector, differentiable, derivative, partial derivative, gradient, curve, tangent vector, Jacobian determinant, continuously differentiable or C^1, locally invertible, partial derivative of order k, k-times continuously differentiable function (C^k), smooth path, piecewise smooth path, closed path, simple path, smooth reparametrization, preserve/reverse orientation, one form, path integral of a one-form (Def 9.2.9), arc length measure, integral with respect to arc length (Def 9.2.14), length of a piecewise smooth path

Comments

- The book defined eigenvalue on page 33, but didn't seem to define eigenvector. Supposing A is an $n \times n$ matrix, a complex number $\lambda \in C$ is an *eigenvector* if there is a non-zero vector $x \in \mathbb{C}^n$ such that $Ax = \lambda x$. Any non-zero vector x such that there is a $\lambda \in \mathbb{C}$ satisfying $Ax = \lambda x$ is called an *eigenvector*.
- The book uses the term "locally invertible" but didn't formally define it. A function between metric spaces $f : X \to Y$ is *locally invertible* at $x \in X$ if there is an open set $U \subset X$ such that $x \in U$, $f(U)$ is open, and $f|_U : U \to f(U)$ is a bijection. Local invertibility is a consequence of the Inverse function theorem.
- The book didn't seem to formally define critical point. A *critical point* of a differentiable function f is an x in the domain such that $Df(x)$ is zero.

Results you should know. Below is a list of results you should know, and be able to apply. Results I think you should be able to prove are in bold. You will be asked to prove at least one of these results on the midterm.

Criterion for a subset of a vector space to be a subspace (Prop 8.1.6), **Span is a subspace (Prop 8.1.11)**, *Vectors have a unique representation in a basis (Prop 8.1.13), Properties of vector spaces related to span, linear independence and bases (Prop 8.1.14), Properties of linear maps (Prop 8.1.16),* **A linear map is determined by its values on a basis in the domain (Prop 8.1.17)**, *A linear map from a finite dimensional space to itself is one-to-one if and only only if it is onto, Arbitrary intersections of convex sets are convex (Prop 8.1.23), Linear maps send convex sets to convex sets (Prop 8.1.25), Cauchy-Schwarz* *inequality (Lemma 7.1.4 and Thm 8.2.2),* **Properties of the operator norm in finite dimensional spaces (Props 8.2.4-5)**, *The map $A \mapsto A^{-1}$ is continuous (Prop 8.2.6), Properties of the determinant (Props 8.2.8-10), Uniqueness of the derivative (Prop 8.3.2),* **Differentiable implies continuous (Prop 8.3.5)**, *The derivative is a linear operator (Prop 8.3.6), Matrix representation of the derivative (Prop. 8.3.9), Bounded derivative on a convex domain implies Lipschitz (Prop 8.4.2), Polynomials are C^1, Inverse function theorem, Implicit function theorem, Swapping order of partial derivatives in C^2 (Prop. 8.6.2), A piecewise smooth reparametrization of a piecewise smooth path is piecewise smooth (***Particularly in the case when both are smooth rather than piecewise smooth***)(Prop 9.2.6), Path integrals of one-forms are independent of parameterization (***especially in the case when both the path and reparameterizations are smooth***)(Prop 9.2.10 and 9.2.12),* **Integrals with respect to arc length are independent of paramterization (Prop 9.2.15)**

Example questions: These are some problems I have written for exams in the past.

1. Are the functions below differentiable at $\mathbf{0}$? Justify your answer with a proof.

$$\mathbf{f}(x,y,z) = \sqrt[3]{xyz} \quad \text{and} \quad \mathbf{g}(x,y,z) = \sqrt[3]{x^2yz}.$$

2. Let $g(a,b,c)$ be a C^2 function $\mathbb{R}^3 \to \mathbb{R}$. Let $f(x,y) = g(x^2, xy, y^2)$.
   (a) Compute $\frac{\partial f}{\partial x}(x,y)$ in terms of x, y and partial derivatives of g.
   (b) Compute $\frac{\partial^2 f}{\partial y \partial x}(x,y)$ in terms of x, y and partial derivatives of g.

3. (a) Complete the following definition. Let $S \subset \mathbb{R}^n$ be an open set. The function $f : S \to \mathbb{R}$ is differentiable at $\mathbf{a} \in S$ if
   (b) Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies $f(0,0) = 0$ and

$$x - x^2 - y^2 \le f(x,y) \le x + x^2 + y^2 \quad \text{for all } (x,y) \in \mathbb{R}^2.$$

   Prove that f is differentiable at $(0,0)$.

4. Show that the two side-lengths of an isosceles triangle can be expressed as a function of the perimeter and area of the triangle locally near any non-equilateral isosceles triangle.

5. Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a differentiable function. Define $g(x,y) = |f(x,y)|$ where $|\cdot|$ denotes the Euclidean norm.
   (a) Give a formula for $\nabla g(x,y)$ at any point (x,y) for which $f(x,y) \neq (0,0)$. Your formula should be in terms of x, y, the entries of $f(x,y)$, and the partial derivatives of the two component functions of f.
   (b) Show that if $Df(x,y)$ is the zero matrix, then (x,y) is a critical point of g. That is, show that $Df(x,y) = 0$ implies that $Dg(x,y) = 0$.

6. Let $\alpha : \mathbb{R} \to \mathbb{R}^3$ be a parameterized C^1 path in the plane $z = 0$ and let $\beta : \mathbb{R} \to \mathbb{R}^3$ be a parameterized C^1 path in the plane $z = 1$. Define

$$F : \mathbb{R}^3 \to \mathbb{R}^3 \quad \text{by} \quad F(s,t,z) = \alpha(s) + z\big(\beta(t) - \alpha(s)\big).$$

Then, when s and t are held constant, the map $z \mapsto F(s,t,z)$ parameterizes the line joining $\alpha(s)$ to $\beta(t)$.

   (a) Use the Inverse Function Theorem to show that if $z_0 \notin \{0,1\}$ and $\alpha'(s_0)$ and $\beta'(t_0)$ are not parallel, then F is invertible in a neighborhood of the point (s_0, t_0, z_0) in the domain.

7. Let $\mathbf{G}(u,v) = (uv - 3, u^2 + 2v)$.

   (a) Suppose V is an open neighborhood of $(-3,6) \in \mathbb{R}^2$ and that $\mathbf{H} : V \to \mathbb{R}^2$ is a map so that $\mathbf{G} \circ \mathbf{H}(x,y) = (x,y)$ for all $(x,y) \in V$. ($\mathbf{H}$ is a local inverse to $\mathbf{G}$ at $(0,3)$.) Find the matrix of partial derivatives $D\mathbf{H}(-3,6)$.

   (b) Let B_ϵ denote the open ball of radius ϵ at $(0,3)$. Estimate the quantity

$$\text{Area } \mathbf{G}(B_\epsilon)/\text{Area } B_\epsilon$$

   for ϵ small. (Concretely, what is the limit of this ratio as $\epsilon \to 0$?)

8. Complete the following definitions:

   (a) The *euclidean norm* of a vector $x \in \mathbb{R}^n$ is ...

   (b) Let (X, d_X) and (Y, d_Y) be metric spaces and $K \in \mathbb{R}$ be a real number with $K \geq 0$. A function $f : X \to Y$ is *K-Lipschitz* if ...

   (c) Let $U \subset \mathbb{R}^n$ be open and let $f : U \to \mathbb{R}^m$ be a function. We say f is *continuously differentiable* if ...

9. Let X and Y be finite dimensional normed vector spaces. Recall that the *operator norm* of $A \in L(X,Y)$ is

$$\|A\| = \sup\{\|Ax\| : \ x \in X \text{ and } \|x\| = 1\}$$

and satisfies $\|Ax\| \leq \|A\|\|x\|$ for all $x \in X$. Show that if both A and B are elements of $L(X,Y)$, then $\|A + B\| \leq \|A\| + \|B\|$.

10. Let $U \subset \mathbb{R}^n$ be open and let $f : U \to \mathbb{R}^m$ be differentiable at $p \in U$. Prove that f is continuous at p.

11. Let $U \subset \mathbb{R}^n$ be open and $p \in U$. Suppose f, g, and h are functions $U \to \mathbb{R}$ such that

$$f(p) = g(p) = h(p) \quad \text{and} \quad f(q) \leq g(q) \leq h(q) \quad \text{for all } q \in U.$$

   (a) Prove that if both f and h are differentiable at p and $\nabla f(p) = \nabla h(p)$, then g is also differentible and $\nabla f(p) = \nabla g(p)$. (Recall that since the codomain of f is $\mathbb{R}$, we have $Df(p)h = \nabla f(p) \cdot h$.)

(b) Show that the hypothesis $\nabla f(p) = \nabla h(p)$ was superfluous: Prove that the other hypotheses (including that f and h are differentiable at p) guarantee that $\nabla f(p) = \nabla h(p)$.

12. Call a C^1 function $g : \mathbb{R} \to \mathbb{R}$ *expanding* if $|g'(x)| > 1$ for all $x \in \mathbb{R}$. Note that this implies by the Mean Value Theorem that

$$|g(b) - g(a)| > |b - a| \quad \text{whenever } a \neq b.$$

Suppose g and h are C^1 expanding functions $\mathbb{R} \to \mathbb{R}$. Define

$$F : \mathbb{R}^2 \to \mathbb{R}^2 \quad \text{by} \quad F(x, y) = \big(x + g(y), y + h(x)\big).$$

(a) Use the Inverse Function Theorem to show that F is locally invertible at all points $(x, y) \in \mathbb{R}^2$.

(b) Show that F is one-to-one.

(c) Show that F is not necessarily surjective by considering the case of $g(y) = -y - e^{-y}$ and $h(x) = -x - e^{-x}$.

13. Let $\gamma : [a, b] \to \mathbb{R}^n$ be a smooth path. Suppose $h : \mathbb{R} \to \mathbb{R}$ is C^1 and $h'(t) > 0$ for every $t \in \mathbb{R}$. Suppose $h(c) = a$, $h(d) = b$ and $c < d$. Then

$$\beta = \gamma \circ h\Big|_{[c,d]} \; : \; [c, d] \to \mathbb{R}^n$$

is a smooth orientation-preserving reparameterization of γ.

Let $\omega = \displaystyle\sum_{j=1}^{n} \omega_j \, dx_j$ be a one-form defined on $\gamma([a, b])$. Prove that $\int_\beta \omega = \int_\gamma \omega$.