Math 32404: Advanced Calculus II: Midterm 1 Review

Material Covered: §7.1 - 7.5, §8.1, §8.2.1, and §8.2.2

Definitions. You will be asked to define several terms on the test. **These terms all have one definition, as given in the book. You are expected to know this definition.** The following is a list of terms which might appear. (Others might appear as well).

metric space, Euclidean metric, subspace metric/topology, bounded set, diameter, open ball, closed ball, open set, closed set, neighborhood, connected, disconnected, clopen, closure, interior, boundary, sequence, bounded sequence, subsequence, converge, limit, convergent, divergent, Cauchy sequence, complete metric space, compact set, sequentially compact, totally bounded, Continuous at a point, continuous function, uniformly continuous, Lipschitz continuous, cluster point, limit of a function (Def 7.5.15), diverges, scalar, vector space, vector, (vector) subspace, linear combination, span, linearly independent, basis, dimension, linear map, invertible linear map, convex subset of a vector space, convex combination, convex hull, norm on a vector space, operator norm

Results you should know. Below is a list of results you should know, and be able to apply. Results I think you should be able to prove are in bold. You will be asked to prove at least one of these results on the midterm.

Cauchy-Schwarz inequality (Lemma 7.1.4 and Thm 8.2.2), **Relations between union and intersection and open/closed sets (Props 7.2.6 and 7.2.8)**, *Open sets in the subspace topology (Prop 7.2.11), Criterion for a set in the subspace topology to be disconnected (Prop 7.2.14), Criterion for a point to lie in the boundary of a set (Prop 7.2.27),* **A convergent sequence is bounded (Prop 7.3.4)**, *Relation between convergence of a sequence and convergence of subsequences (Prop 7.3.6), Convergence in $\mathbb{R}^n$ (Prop 7.3.9), Relation between convergence and topology (Prop 7.3.11),* **Limit of a convergent sequence in a closed set is in the set (Prop 7.3.12)**, **Convergent sequences are Cauchy (Prop 7.4.2)**, **Compact sets are closed and bounded (Prop 7.4.9)**, *Lebesgue covering lemma (Lemma 7.4.10), A metric space is compact if and only if it is sequentially compact (Theorem 7.4.11),* **Closed subsets of compact sets are compact (Prop 7.4.13)**, *Heine-Borel Theorem (Thm 7.4.14), A metric space is compact if and only if it is complete and totally bounded (covered in class, see also Exercise 7.4.12), Relation between continuity and sequences (Prop 7.5.2),* **Image of a compact set under a continuous map is compact (Lemma 7.5.5)**, *Continuous functions to $\mathbb{R}$ attain their minima and maxima on compact sets (Theorem 7.5.6), Relations between continuity and topology (Lemma 7.5.7, Thm 7.5.8, class), A continuous function with compact domain is uniformly continuous (Theorem 7.5.11), Characterization of continuity in terms of limits (Lemma 7.5.17), Criterion for a subset of a vector space to be a subspace (Prop 8.1.6),* **Span is a subspace (Prop 8.1.11)**, *Vectors have a unique representation in a basis (Prop 8.1.13), Properties of vector spaces related to span, linear independence and bases (Prop 8.1.14), Properties of linear maps (Prop 8.1.16),* **A linear map is determined by**

its values on a basis in the domain (Prop 8.1.17), *A linear map from a finite dimensional space to itself is one-to-one if and only only if it is onto, Arbitrary intersections of convex sets are convex (Prop 8.1.23), Linear maps send convex sets to convex sets (Prop 8.1.25),*

Example questions: These are some problems I have written for exams in the past.

1. Let X be a metric space and $A \subset X$ be a non-empty set. Define

$$f : X \to \mathbb{R}; \quad x \mapsto \inf \{d(x, a) : \ a \in A\}.$$

   Prove that f is a continuous function.

2. Let (S, d) be a metric space. Suppose $\{s_n \in S\}$ be a sequence converging to $s \in S$. Suppose $\{t_n \in S\}$ is a sequence of points such that $d(s_n, t_n) < \frac{1}{n}$ for all n. Prove that $\{t_n\}$ converges to s.

3. Recall a function $f : X \to Y$ is *surjective* (or *onto*) if for all $y \in Y$ there is an $x \in X$ so that $f(x) = y$.

   Let $B = \{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 < 1\}$ and let $\overline{B}$ be its closure, which is a closed ball. Prove that there is no continuous surjective map $\mathbf{f} : \overline{B} \to B$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. The *graph of* f is the set

$$G = \big\{(x, y) \in \mathbb{R}^2 \ : \ y = f(x)\big\}.$$

   Prove that G is closed.

5. Let $S = \{(x, y) \in \mathbb{R}^2 \ : \ (x, y) \neq \mathbf{0}\}$. Prove that any continuous function $f : S \to \mathbb{R}$ satisfying the equation

$$f(-x, -y) = -f(x, y) \quad \text{for all } (x, y) \in S$$

   has a zero. That is, prove that there is a point $(x_0, y_0) \in S$ with $f(x_0, y_0) = 0$.

6. Let $f : \mathbb{R}^2 \smallsetminus \{(0, 0)\} \to \mathbb{R}$ denote the function

$$f(x, y) = \frac{|x| - |y|}{|x| + |y|}.$$

   Prove that $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist.

7. Give counterexamples to the following false statements:
   (a) If $F \subset \mathbb{R}^2$ is closed and $\{\mathbf{v}_k\}$ is a sequence in F, then the sequence $\{\mathbf{v}_k\}$ has a convergent subsequence.
   (b) If $K \subset \mathbb{R}^2$ is a compact set, $f : K \to \mathbb{R}$ is continuous, and there are $\mathbf{v}, \mathbf{w} \in K$ such that $f(\mathbf{v}) < 0 < f(\mathbf{w})$, then there is an $\mathbf{x} \in K$ such that $f(\mathbf{x}) = 0$.

(c) If $E \subset \mathbb{R}^2$ is a bounded set, then every point in ∂E lies in $\overline{E} \setminus E$. (Here ∂E denotes the boundary of E, and $\overline{E}$ represents the closure of E.)

8. Let K be a compact subset of a metric space X with distance function d. Use the definition of compactness to prove that that for any $x_0 \in X$, there is an $r > 0$ such that $K \subset B(r, x_0)$.

9. Let X be a metric space with distance function $d : X \times X \to \mathbb{R}_{\geq 0}$. Let a and b be arbitrary points of X. Prove that the set

$$U = \{x \in X : \; d(a, x) + d(b, x) < 1\}$$

is open.

10. Complete the following definitions:

(a) Let (X, d) be a metric space. A subset $S \subset X$ is *bounded* if ...

(b) Let (X, d) be a metric space and $A \subset X$. The *closure* of A is the set ...

(c) Let (X, d_X) and (Y, d_Y) be metric spaces. Then $f : X \to Y$ is *uniformly continuous* if ...

(d) A subset U of a vector space is *convex* if ...

11. Let (S, d) be a metric space. Suppose $\{s_n \in S\}$ be a sequence converging to $s \in S$. Suppose $\{t_n \in S\}$ is a sequence of points such that $d(s_n, t_n) < \frac{1}{n}$ for all n. Prove that $\{t_n\}$ converges to s.

12. Let (X, d) be a metric space, and assume that $E_1 \subset X$ is a compact set. Suppose that $E_1 \supset E_2 \supset E_3 \supset \cdots$ is a nested sequence of nonempty closed subsets of X. Prove that $\bigcap_{j=1}^{\infty} E_j$ is nonempty.

13. Let X be a vector space. Recall that a *norm* on X is a function $\|\cdot\| : X \to \mathbb{R}$ such that

   1. $\|\mathbf{x}\| \geq 0$, with $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}_X$.
   2. $\|c\mathbf{x}\| = |c|\|\mathbf{x}\|$ for all $c \in \mathbb{R}$ and $\mathbf{x} \in X$.
   3. $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in X$. (triangle inequality)

Prove that the sum of two norms satisfies the triangle inequality. That is, prove that if $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms on a vector space X, then the function $\|\cdot\|_3$ defined by

$$\|\mathbf{x}\|_3 = \|\mathbf{x}\|_1 + \|\mathbf{x}\|_2$$

satisfies statement (3) above.

14. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} \frac{y}{x} & \text{if } x \neq 0 \\ y & \text{if } x = 0. \end{cases}$$

(a) Prove that f is not continuous at $(0,0)$.

(b) Let $D = \{(x,y) \in \mathbb{R}^2 : \ |y| \leq x^2\}$. Let g be the *restriction* of f to D. That is, g is only defined when $(x,y) \in D$ and $g(x,y) = f(x,y)$ when $(x,y) \in D$. Prove that g is continuous at $(0,0)$.

15. What follows is a rephrased version of a proposition in the book:

Proposition 8.1.6. Let S be a subset of a vector space X. Then S is a subspace of X if the following statements hold:

1. $\mathbf{0}_X \in S$.
2. S is closed under addition: If $\mathbf{x}, \mathbf{y} \in S$, then $\mathbf{x} + \mathbf{y} \in S$.
3. S is closed under scalar multiplication: If $\mathbf{x} \in S$ and $a \in \mathbb{R}$, then $a\mathbf{x} \in S$.

Now suppose $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ is a finite subset of a vector space X. Prove that

$$S = \operatorname{span}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$$

is a subspace of X.