Math 32404: Advanced Calculus II: Final Exam Review

Material Covered: §7.1-7.5, 8.1-8.5, 9.1-9.3, 10.1-10.6. Material since the last midterm (§9.1-9.3 and 10.1-10.6) will be emphasized (approximately 1/2 to 2/3 of the exam).

Definitions. You will be asked to define several terms on the test. You have already been given definition lists in the Midterm 1 and Midterm 2 reviews. The following are from sections 9.1-9.3 and 10.1-10.6 and might appear on the exam (in addition to those listed in the other reviews). **These terms normally have one definition, as given in the book. You are expected to know this definition. Exceptions are in the comments below.** The following is a list of terms which might appear.

smooth path, piecewise smooth path, closed path, simple path, smooth reparametrization, preserve/reverse orientation, one form, path integral of a one-form (Def 9.2.9), arc length measure, integral with respect to arc length (Def 9.2.14), length of a piecewise smooth path, path independent, path connected, star-shaped, closed rectangle, open rectangle, n-dimensional volume of a rectangle, partition, subrectangle, lower and upper Darboux sums, lower and upper Darboux integrals, refinement, integrable, Riemann integral, support of a function, compact support, outer measure, measure zero, null set, characteristic function, Jordan measurable set, volume of a Jordan measurable set, Riemann integral of a function on a bounded Jordan measurable set (and Riemann integrable in the same context), bounded domain with piecewise smooth boundary, positively oriented, negatively oriented,

Comments

- We used a different definition for path connected: A subset Y of a metric space X is *path connected* if for every pair of points $y_0, y_1 \in Y$, there is a continuous $\gamma : [0,1] \to Y$ such that $\gamma(0) = y_0$ and $\gamma(1) = y_1$. (Such a γ is a *path*.)

Results you should know. Below is a list of results you should know and be able to apply. These are from §9.1-9.3 and §10.1-10.6. (Again you also know the results that appear in the other review sheets.) Results I think you should be able to prove are in bold. You will be asked to prove at least one of these results on the exam.

*Leibniz integral rule (Thm 9.1.1), A piecewise smooth reparametrization of a piecewise smooth path is piecewise smooth (***Particularly in the case when both are smooth rather than piecewise smooth***)(Prop 9.2.6), Path integrals of one-forms are independent of parameterization (***especially in the case when both the path and reparameterizations are smooth***)(Prop 9.2.10 and 9.2.12),* **Integrals with respect to arc length are independent of paramterization (Prop 9.2.15)***, Integration of a one-form ω is path independent if and only if $\omega = df$ (Prop 9.3.3), Integration of a one-form ω is path independent if and only if the integral over every piecewise smooth closed path is zero (Prop 9.3.4), Poincaré Lemma (Thm 9.3.6), Relations between upper and lower Darboux sums (Prop 10.1.2), Relationship between refinement and upper and* *lower sums (Prop 10.1.5), The upper integral can not be less than the lower integral (Prop 10.1.6), Bounds on the integral (Prop. 10.1.8), Linearity of the integral (Prop 10.1.10),* **Monotonicity of the integral (Prop. 10.1.11)**, **Criterion for integrability (Prop. 10.1.12)**, *If a function is integrable over a rectangle, then it is also integrable over any subrectangle (Prop 10.1.13),* **Continuous functions are integrable (Thm 10.1.15)**, *Integral of a function with compact support (Prop 10.1.19), Fubini's Theorem (Thms 10.2.2 and 10.2.3),* **A countable union of measure zero sets is measure zero (Prop 10.3.4)**, *The C^1 image of a measure zero set has measure zero (PRop 10.3.10), Lebesgue-Vitali Theorem (Thm 10.4.3), Algebraic properties of integrable functions (Corollary 10.4.4), Criterion for a set to be Jordan measurable (Prop 10.5.1), Basic properties of the Riemman integral on Jordan Measurable sets (Props 10.5.6, 10.5.7, 10.5.8), Bounded domains with piecewise smooth boundary are Jordan Measurable (Prop 10.6.3), Green's Theorem (Thm 10.6.4),*

Example questions:

1. (a) State Fubini's Theorem for a real-valued function f defined on the rectangle

$$R = \{(x, y) \ : \ a \leq x \leq b \text{ and } c \leq y \leq d\}.$$

(You only need to state and give hypotheses for one of the two integral formulas.)

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. For $s > 0$ and $t > 0$, let $R_{s,t}$ be the rectangle

$$R_{s,t} = \{(x, y) \in \mathbb{R}^2 \ : \ 0 \leq x \leq s \text{ and } 0 \leq y \leq t\}.$$

Define $g(s, t) \in \mathbb{R}$ for $s > 0$ and $t > 0$ according to the rule

$$g(s, t) = \iint_{R_{s,t}} f(x, y) \, dA.$$

Prove that the partial derivative of g with respect to s is given by

$$\frac{\partial g}{\partial s}(s, t) = \int_0^t f(s, y) \, dy.$$

(*Hint:* Use the Fundamental Theorem of Calculus.)

2. Consider the coordinate change from Polar to Cartesian coordinates,

$$\mathbf{H}(r, \theta) = (r \cos\theta, r \sin\theta).$$

Recall that if W is a region in the (r, θ)-plane, and the map H restricts to a one-to-one map of W onto its image $\mathbf{H}(W)$ then

$$\text{Area}\big(\mathbf{H}(W)\big) = \iint_W r \, dr \, d\theta.$$

Find functions $P(r,\theta)$ and $Q(r,\theta)$ so that

$$\text{Area}\big(\mathbf{H}(W)\big) = \int_{\partial W} P\ dr + Q\ d\theta.$$

Explain why your answer is correct. (*Hint:* Apply Green's theorem in the (r,θ)-plane.)

3. Let $\omega(x,y) = \frac{-y}{x^2+y^2}\ dx + \frac{x}{x^2+y^2}\ dy$. Note that the one-form ω is not defined at the origin. Also note that if P and Q are such that $\omega = P\ dx + Q\ dy$ then $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$ on the domain of definition of ω.

   Let C be a circle in the plane oriented counter-clockwise that does not pass through the origin. The curve C bounds a disk D. It is true that

$$\int_C \omega = \begin{cases} 2\pi & \text{if } D \text{ contains the origin,} \\ 0 & \text{if } D \text{ does not contain the origin.} \end{cases} \tag{1}$$

   (We discussed similar facts in class.)

   (a) For a point $p = (a,b)$ define

$$\eta_p(x,y) = \omega(x-a, y-b).$$

   Give a characterization similar to (1) for line integrals of the one-form η_p over circles in the plane. Explain why your characterization is correct.

   (b) Suppose $p_1, \ldots, p_n$ is a list of points in the plane. Define a new one-form

$$\alpha(x,y) = \eta_{p_1}(x,y) + \eta_{p_2}(x,y) + \ldots + \eta_{p_n}(x,y).$$

   If C is a circle in the plane oriented counterclockwise which does not intersect any of the points $p_1, \ldots, p_n$, describe how to determine $\int_C \alpha$. Explain why your method for determining this line integral is correct.

4. Suppose $\gamma : \mathbb{R} \to \mathbb{R}^2$ is C^1 parameterization satisfying of a curve in the plane. Suppose $\gamma(t) = \gamma(t + 2\pi)$ for all $t \in \mathbb{R}$ (guaranteeing that the curve is closed) and $\gamma'(t) \neq \mathbf{0}$ for all $t \in \mathbb{R}$. Define $F : \mathbb{R}^2 \to \mathbb{R}$ by

$$F(s,t) = \|\gamma(s) - \gamma(t)\|^2.$$

   Prove that (s,t) is a critical point for F if and only if $\gamma(s) = \gamma(t)$ or both $\gamma'(s)$ and $\gamma'(t)$ are orthogonal to $\gamma(s) - \gamma(t)$.

5. For a real number $K \geq 0$, a function $g : [0,1] \to \mathbb{R}^2$ is *K-Lipschitz* if

$$\|g(s) - g(t)\| \leq K|s - t|$$

   for all $s, t \in [0,1]$. Prove that the image $g([0,1])$ of a K-Lipschitz function has zero measure.

6. Let $R = [a_1, b_1] \times \ldots \times [a_n, b_n]$ be a rectangle in $\mathbb{R}^n$. Let f and g be functions $R \to \mathbb{R}$ such that $f(x) \le g(x)$ for all $x \in R$.
   (a) Prove that for any partition $\mathcal{P}$ of R, we have $L(\mathcal{P}, f) \le U(\mathcal{P}, g)$.
   (b) Prove that $\underline{\int_R} f \le \overline{\int_R} g$.
   (c) Prove that if f and g are integrable, then $\int_R f \le \int_R g$.

7. (a) Let $S \subset \mathbb{R}^n$ be a compact set, and $f : S \to \mathbb{R}$ be a continuous function such that $f(\mathbf{x}) > 0$ for all $\mathbf{x} \in S$. Prove that there is a $c > 0$ for which $f(\mathbf{x}) \ge c$ for all $\mathbf{x} \in S$.
   (b) Give an example of a bounded $S \subset \mathbb{R}^n$ for some n, and a continuous function $f : S \to \mathbb{R}$ such that $f(\mathbf{x}) > 0$ for all $\mathbf{x} \in S$ but $\inf\{f(\mathbf{x}) : \mathbf{x} \in S\} = 0$.

8. Consider the one-form on $\mathbb{R}^2$ given by the equation

$$\omega(x, y) = (2x + \sin y)\ dx + (-1 + x \cos y)\ dy.$$

   (a) Are integrals of ω path independent on $\mathbb{R}^2$? If they are, find a function f such that $\omega = df$.
   (b) Consider the curve C parameterized by

$$g(t) = \big(2^t \cos(t\pi), 5^t \sin(t\pi)\big) \quad \text{for } 0 \le t \le 2.$$

   Evaluate $\int_C \omega$.

9. Complete the following definitions.
   (a) If (X, d_X) and (Y, d_Y) are metric spaces, a function $f : X \to Y$ is *uniformly continuous* if ...
   (b) Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^m$ be a function. We say f is *differentiable* at $x \in U$ if ...
   (c) Given a partition $\mathcal{P}$ of a rectangle $R \subset \mathbb{R}^n$ and a bounded function $f : R \to \mathbb{R}$, the *lower* and *upper Darboux sums* are ...
   (d) The *lower* and *upper Darboux integrals* of f over R are ...
   (e) Let $S \subset \mathbb{R}^n$ be bounded and Jordan measurable and let $f : S \to \mathbb{R}$ be a bounded function. The Riemann integral of f over S is ...

10. (a) Recall that the *outer measure* of a set $S \subset \mathbb{R}^n$ is defined to be

$$m^*(S) = \inf \sum_{R \in \mathcal{R}} V(R),$$

   where the infinimum is taken over all countable collections of rectangles $\mathcal{R}$ such that $S \subset \bigcup_{R \in \mathcal{R}} R$, where we are allowing both the sum and the infimum to be ∞. A *null set* is a set S for which $m^*(S) = 0$.
   Prove that if S_i is a null set for each $i \in \mathbb{N}$, then $U = \bigcup_{i=1}^{\infty} S_i$ is also a null set.

(b) Recall that a bounded set $A \subset \mathbb{R}^n$ is Jordan measurable if and only $m^*(\partial A) = 0$. If $\{A_j\}_{j=1}^{\infty}$ is a sequence of Jordan measurable sets such that $U = \bigcup_{j=1}^{\infty} A_j$ is bounded, is it true that U is also Jordan measurable? Either prove this statement correct or produce a counterexample.

11. Let $\gamma : [a, b] \to \mathbb{R}^n$ be a smooth path. Suppose $h : \mathbb{R} \to \mathbb{R}$ is C^1 and $h'(t) > 0$ for every $t \in \mathbb{R}$. Suppose $h(c) = a$, $h(d) = b$ and $c < d$. Then

$$\beta = \gamma \circ h\Big|_{[c,d]} \ : \ [c, d] \to \mathbb{R}^n$$

is a smooth orientation-preserving reparameterization of γ.

Let $\omega = \sum_{j=1}^{n} \omega_j \, dx_j$ be a one-form defined on $\gamma([a, b])$. Prove that $\int_\beta \omega = \int_\gamma \omega$.

12. Suppose that $R \subset \mathbb{R}^n$ is a closed rectangle and $f : R \to \mathbb{R}$ is Riemann integrable over R. Suppose $g : f(R) \to \mathbb{R}$ is continuous and that $g \circ f(R)$ is a bounded subset of $\mathbb{R}$. Prove that $g \circ f$ is Riemann integrable over R.

13. Let (X, d) be a metric space and $K \subset X$. Recall that K is *compact* if for every collection $\mathcal{U}$ of open subsets of X such that

$$K \subset \bigcup_{U \in \mathcal{U}} U,$$

there is a finite subcollection $\mathcal{F} \subset \mathcal{U}$ such that $K \subset \bigcup_{U \in \mathcal{F}} U$.

(a) Recall that a subset $S \subset X$ is *bounded* if there is a $p \in X$ and an $r > 0$ such that S is contained in the closed ball of radius r centered at p.

Prove that a compact subset of a metric space is bounded. (*Hint:* Cover by balls of radius $n \in \mathbb{N}$.)

(b) Recall that a subset $U \subset X$ is *open* if for every $p \in U$, there is an $r > 0$ such that the open ball centered at p of radius r, $B(p, r)$, is contained in U. Also recall that a subset $S \subset X$ is *closed* if the complement $X \setminus S$ is open.

Prove that a compact subset of a metric space is closed. (*Hint:* Cover by complements of closed balls of radius $\frac{1}{n}$ with $n \in \mathbb{N}$.)

14. Let X be a vector space. Recall that a *norm* on X is a function $\|\cdot\| : X \to \mathbb{R}$ such that the following statements hold:

- $\|x\| \geq 0$, with $\|x\| = 0$ if and only if $x = 0$.
- $\|cx\| = |c|\|x\|$ for all $c \in \mathbb{R}$ and $x \in X$.
- $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$ (*triangle inequality*).

(a) Also recall that a set $A \subset X$ is *convex* if $x, y \in A$ and $t \in [0, 1]$ implies that $(1 - t)x + ty$ is in A. Prove that for any norm on any vector space, the closed unit ball

$$B = \{x \in X : \ \|x\| \leq 1\}$$

is convex.

(b) Recall that a sequence $\{x_n\}$ is a vector space *converges* to x if

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

Prove that for any norm on any vector space, if $\{x_n \in X\}_{n=1}^{\infty}$ is a sequence coverging to $x \in X$, and $\{c_n \in \mathbb{R}\}_{n=1}^{\infty}$ is a sequence of real numbers converging to $c \in \mathbb{R}$, then the sequence $\{c_n x_n \in X\}_{n=1}^{\infty}$ converges to cx. (*Remark:* This will prove that scalar multiplication, viewed as the function $\mathbb{R} \times X \to X$ and defined by $(c, x) \mapsto cx$, is continuous.)