INDUCTION ON TREES AND GRAPHS

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The goal of these notes are to give some more examples of rigorous inductive arguments involving trees and graphs, following [Lev21]. The following result is crucial for arguments involving induction on trees:

Proposition 1 ([Lev21, Prop. 4.2.3])**.** *If a tree has at least two vertices, then it has at least two vertices of degree one.*

A vertex of a trees with degree one is also called a *leaf.* Removing a leaf and the incident edge leads to a tree with one fewer vertex (because the resulting subgraph is clearly still connected and doesn't have cycles).

The book gives a good careful inductive proof of the statement that all trees satisfy $v = e + 1$, where v is the number of vertices and e is the number of edges [Lev21, Prop. 4.2.4]. Levin uses rooted trees to prove the following result, but we will use induction.

Proposition 2. *Every tree is bipartite.*

Recall that a graph is bipartite if its vertex set can be split into two disjoint sets A and B and every edge has the form $\{a, b\}$ where $a \in A$ and $b \in B$. (That is, there are no edges joining vertices in the same set.)

Inductive Proof. For each integer $n \geq 1$, let $P(n)$ be the statement that every tree with n vertices is bipartite. To prove this statement by induction we need to prove both:

- $P(1)$. That is, we need to prove every tree with one vertex is bipartite.
- For every $k \geq 1$, $P(k) \implies P(k+1)$. That is, we can assume that if every tree with k vertices is bipartite, and we need to prove that every tree with $k+1$ vertices is bipartite.

Base case: We will show that every tree with one vertex is bipartite. Let T be a tree with 1 vertex. Then we can place that vertex in set A and define B to be the empty set. Since there are no edges, this tree is bipartite.

Inductive step: Assume every tree with k vertices is bipartite. We will show that every tree with $k+1$ vertices is bipartite. Let T be a tree with $k+1$ vertices. Since $k+1 \geq 2$, we know that T has at least two leaves. Choose one leaf and call it v. Let w be the other endpoint of the single edge of the leaf v. Let T' be the subgraph obtained by removing v and the edge $\{v, w\}$. Then T' is a tree with one fewer vertex, so by our inductive assumption T' is bipartite. This means we can split the vertices of T' into two subsets, $A' \cup B'$, and all edges have the form $\{a', b'\}$ with $a' \in A'$ and $b' \in B'$.

We need to write the vertices of T as a union of two sets with all edges going between vertices from different sets. Note that w is a vertex of T'. To show T is bipartite, we break into two cases. First suppose $w \in A'$. Then we define

$$A = A' \quad \text{and} \quad B = B' \cup \{v\}.$$

Date: April 28, 2022.

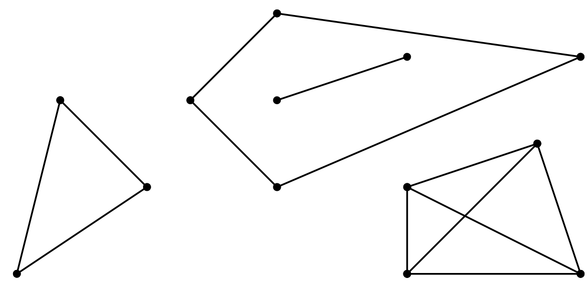

FIGURE 1. A graph with four components.

Then all the vertices are in $A \cup B$. To see this makes T bipartite, consider any edge e of T. If e is an edge of T' then e has one vertex in A' and one in B', so it also has one in A and one in B. If e is not an edge of T', then e must be the edge $\{v, w\}$ and $v \in B$ and $w \in A$. This shows T is bipartite. The second case is when $w \in B'$. In this case we define

$$A = A' \cup \{v\} \quad \text{and} \quad B = B'.$$

Then again any edge of T is either an edge of T', which has one vertex in A and one in B like before, or is the edge $\{v, w\}$ and here we have $v \in A$ and $w \in B$. So again T is bipartite. ☐

Remark 3. *Instead of breaking into two different cases in the last paragraph above, we could have said "To show T is bipartite, assume without loss of generality that $w \in A'$" and worked through only the first case. This expression is just to indicate that there are multiple cases, but there is a symmetry (or the objects can be relabeled) to see that the multiple cases work in exactly the same way. Here, the names we are using for the two sets A' and B' are arbitrary, so if $w \in B'$, then we could swap the names of the sets A' and B' to get that $w \in A'$. You can see that the second case is really the same as the first case but with the roles of A' and B' swapped.*

Now recall that if G is a graph a *spanning tree* is a subgraph that is a tree and includes all vertices of G. We'll give a formal proof of:

Theorem 4. *Every connected graph has a spanning tree.*

We'll give an inductive proof of this. The main idea is given a graph G we can pick any edge e of G and remove just that edge (leaving the endpoints). This results in a subgraph G' with one fewer edge. But the graph G' may not be connected. So, this is a problem for us running a simple inductive argument like the inductive argument like the one above.

We need some new terminology. If v_1 is a vertex of the graph G', then the *component* of G' containing v_1 is the subgraph consisting of all vertices and edges that can be reached by paths passing through v_1. Each component is connected (because each vertex of the component has a path to v_1). See Figure 1 for an example. A graph is connected if and only if it has only one component.

Removing an edge from a connected graph can result in either a connected graph or a graph with two components (the components of the two endpoints). See Figure 2.

We will apply our inductive hypothesis to the components a graph G with an edge removed. This subgraph G' is either connected, in which case G' has one fewer edge than G, or G' consists of two components and the sum of the edges of the two components is one fewer than the number of edges of G.

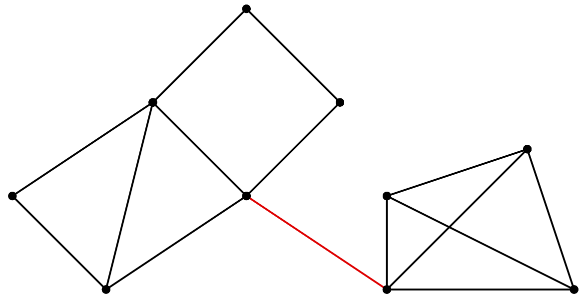

FIGURE 2. The graph here is connected, but removing the red edge results in a subgraph with two components.

Inductive Proof of Theorem 4. Let $P(n)$ be the statement "every connected graph with n edges has a spanning tree." We'll prove that $P(n)$ is true for all $n \geq 0$ using strong induction. To do this we will show:

- $P(0)$ is true.
- For every $k \geq 0$, if $P(i)$ is true for all i with $0 \leq i \leq k$, then then $P(k+1)$ is true.

Base case: Suppose our connected graph G has zero edges. Then it must consist of a single vertex. A graph with a single vertex and no edges is a tree, so G is its own spanning tree.

Inductive step: Let $k \geq 0$. Assume that every connected graph with k or fewer edges has a spanning tree. We'll show every connected graph with $k+1$ edges has a spanning tree. Let G be a graph with $k+1$ edges. We'll prove G has a spanning tree.

Pick any edge e of G. Let G' be the subgraph formed by removing e from G and keeping all vertices and all other edges. Then G' has k edges. As noted above, G' either is connected or consists of two components. We break into these cases below.

First suppose G' is connected. Then from our inductive hypothesis, G' has a spanning tree T'. This means that T' is a subgraph which is a tree that includes all the vertices of G'. Then T' is also a subgraph of G and has all the vertices of G, so T' is also a spanning tree for G.

Now suppose that G' consists of two components. Call these components G_1' and G_2'. Let E_1' and E_2' denote the edge sets of these two components, respectively. Then $|E_1'|+|E_2'| = k$, so both graphs have k or fewer edges. Therefore by inductive hypothesis we can find a spanning tree T_1' for G_1' and a spanning tree T_2' for G_2'. Now define T to be the subgraph of G consisting of T_1', T_2' and the edge e we removed. This subgraph T has all the vertices of G because every vertex of G is either a vertex of G_1' or of G_2' and T_1' and T_2' contain all the vertices of the components containing them. We also claim that T is a tree. The subgraph T is connected, because both T_1' and T_2' are connected and the edge e we added allows us to move from T_1' back and forth to T_2'. Any cycle in T would have to be contained in T_1' (which is impossible because T_1' is a tree), be contained in T_2' (which is likewise impossible), or as we travel around the cycle we travel from T_1' to T_2' and then back again to T_1'. But the only way to travel from T_1' to T_2' or back is through the edge e and cycles can't repeat edges, so this third possibility is also impossible. This proves that T is a tree and a spanning tree for G. $\square$

References

[Lev21] Oscar Levin, *Discrete mathematics: An open introduction*, 3 ed., 2021, http://discrete.openmathbooks.org.

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