GENERATING FUNCTIONS

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These notes follow material in [Lev21, §5.1]. The goal is to summarize how generating functions relate functions with sequences.

Common generating functions. Below we show some examples of common sequences and their generating functions.

Sequence	**Generating function**
Any sequence $\{a_n\}_{n\geq 0}$ with indices starting at zero.	The power series $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$.
A sequence $\{a_n\}_{n\geq 0}$ for which $a_n = 0$ when $n > d$.	The polynomial $a_0 + a_1 x + a_2 x^2 + \ldots + a_d x^d$.
The constant sequence $0, 0, 0, 0, 0, \ldots$.	The constant zero function.
The constant sequence $1, 1, 1, 1, 1, \ldots$.	The function $\frac{1}{1-x}$.
The constant sequence $c, c, c, c, c, \ldots$ for some $c \in \mathbb{R}$.	The function $x \mapsto \frac{c}{1-x}$.
The geometric sequence $1, c, c^2, c^3, \ldots$ for $c \in \mathbb{R}$.	The function $x \mapsto \frac{1}{1-cx}$.
The geometric sequence $r, rc, rc^2, rc^3, \ldots$ for $c, r \in \mathbb{R}$.	The function $x \mapsto \frac{r}{1-cx}$.
The arithmetic sequence $1, 2, 3, 4, 5, \ldots$.	The function $\frac{1}{(1-x)^2}$.
The sequence $a_n = \binom{n+k}{k}$ for an integer $k \geq 0$.	The function $\frac{1}{(1-x)^{k+1}}$.

Operations. Below we assume the sequence $\{a_n\}_{n\geq 0}$ has generating function $A(x)$ and $\{b_n\}_{n\geq 0}$ has generating function $B(x)$. We describe how basic operations with sequences correspond to operations on functions.

Sequence Operation	**Function Operation**
The scalar product sequence $b_n = s a_n$ for some $s \in \mathbb{R}$.	The scaled function: $B(x) = sA(x)$.
The termwise sum of sequences, $c_n = a_n + b_n$.	The sum of the functions: $C(x) = A(x) + B(x)$.
The shifted sequence $0, a_0, a_1, a_2, \ldots$	Multiplication by x gives the function $x \mapsto xA(x)$.
The spaced out sequence $a_0, 0, a_1, 0, a_2, 0, a_3, 0, \ldots$	Substitution of x^2 for x, $x \mapsto A(x^2)$.
The sequence $b_n = a_n c^n$ for some $c \in \mathbb{R}$.	Substitution of cx for x, $B(x) = A(cx)$.

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The sequence $c_n = \sum_{i=0}^{n} a_i b_{n-i}$.	Multiplication: $C(x) = A(x)B(x)$
The sequence of partial sums $d_n = \sum_{i=0}^{n} a_i$.	Division by $1-x$: $D(x) = \dfrac{A(x)}{1-x}$.
The sequence $b_n = (n+1)a_{n+1}$.	The derivative, $B(x) = \frac{d}{dx}A(x)$.

References

[Lev21] Oscar Levin, *Discrete mathematics: An open introduction*, 3 ed., 2021, http://discrete.openmathbooks.org.

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