EULER PATHS

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These notes follow material in [Lev21, §4.5]. Their goal is to give a rigorous proof of the main result of the section on Euler paths and circuits:

Theorem 1 (Euler Paths and Circuits [Lev21, §4.5])**.**

(1) A connected graph has an Euler circuit if and only if the degree of every vertex is even.

(2) A connected graph has an Euler path if and only if there are at most two vertices with odd degree.

A side goal is to see an inductive argument in a more challenging context.

Remark 2 (Connectivity)**.** *The book* [Lev21] *is a bit sloppy and doesn't seem to discuss connectivity in this section. But, it is clear that a graph that is not connected can not have an Euler path or an Euler circuit. As we will see connectivity is important to think about if you want to prove a statement like this, because not all graphs are connected, and you need to be careful with inductive arguments because of this. A graph that is not connected is called* disconnected*.*

These are if and only if statements. So to prove each we need to prove two implications in each case:

- The easier directions are when a graph has an Euler circuit or path, then you can say something about the degrees.
- The harder directions are when you know the degrees of the vertices and you need to produce a Euler circuit or path.

The reason the first directions are "easier" is because you have more structure provided by the Euler circuit or path. Thinking carefully about these will lead us to a proof. To prove the harder implications, we will use an inductive argument to produce an Euler circuit or graph.

1. The easier implications

The following gives one implication for statement (1):

Proposition 3. *If G has an Euler circuit, then the degree of every vertex is even.*

Before we prove this, recall that Euler paths and Euler circuits are examples of walks. We need some notation to work with walks. A walk can be represented by the sequence of vertices visited, $w_0, w_1, \ldots, w_n$. For example, consider the walk below (which happens to be an Euler circuit):

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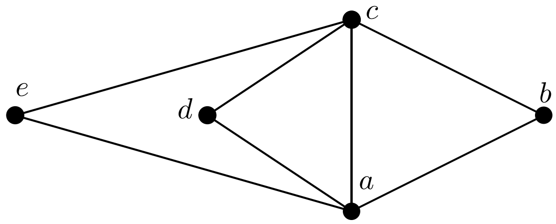
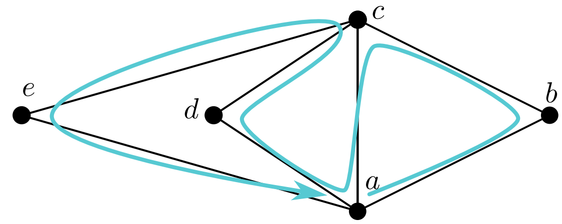




This walk is given by the vertex sequence

$$w_0 = a,\ w_1 = b,\ w_2 = c,\ w_3 = a,\ w_4 = d,\ w_5 = c,\ w_6 = e,\ w_7 = a.$$

To see that this is an Euler cycle, observe that it starts and ends at the same vertex because $w_0 = w_7$ and every edge has the form $\{w_i, w_{i+1}\}$ for exactly one i with $0 \leq i \leq 6$.

Proof of Proposition 3. Suppose G has an Euler circuit. Denote the vertex sequence by $w_0, w_1, \ldots, w_n$. Because it is an Euler circuit, $w_0 = w_n$ and every edge has the form $\{w_i, w_{i+1}\}$ for some i. Fix a vertex v. Observe that the degree of v is given by the formula

$$d(v) = 2\Big|\{i :\ 0 \leq i \leq n-1 \text{ and } w_i = v\}\Big|.$$

(This denotes twice the number of indices less than n for which $w_i = v$.) Observe that if $1 \leq i \leq n-1$ and $w_i = v$, then both the edges $\{w_{i-1}, w_i\}$ and $\{w_i, w_{i+1}\}$ are edges with a vertex of v. Since $w_0 = w_n$ if $w_0 = v$, then the edges $\{w_0, w_1\}$ and $\{w_{n-1}, w_n\}$ both end at v. Because it is an Euler circuit, this accounts for all edges, so $d(v)$ must always be even. $\square$

Now we prove the "easy direction" of statement (2) of Theorem 1.

Proposition 4. *If G has an Euler path, then either the path is an Euler circuit (i.e., the path starts and ends at the same point) or the path starts and ends at different vertices of odd degree and every other vertex has even degree. In particular, if G has an Euler path, then there are at most two vertices with odd degree.*

Proof. Suppose G has an Euler path. Denote the vertex sequence by $w_0, w_1, \ldots, w_n$.

The Euler path is an Euler circuit if $w_0 = w_n$. In this case Proposition 3 tells us that all vertices have even degree. So, there are no vertices of odd degree.

Now suppose that our Euler path is not an Euler circuit. Then $w_0 \neq w_n$. In this case, the degree formula is

$$d(v) = 2\Big|\{i :\ 1 \leq i \leq n-1 \text{ and } w_i = v\}\Big| + \Big|\{i :\ i \in \{0, n\} \text{ and } w_i = v\}\Big|. \tag{1}$$

The last term appears because now if $v = w_0$, then the only edge of our graph that we learn has a vertex of v is $\{w_0, w_1\}$. Similarly if $v = w_n$, then the only edge we learn has a vertex of v is $\{w_{n-1}, w_n\}$. Observe that the first term in our expression for $d(v)$ is always even, and the second term takes the value zero or one. The only time it takes the value one is when $v = w_0$ or $v = w_n$. So, the path must start and end at vertices of odd degree and all other vertices must have even degree. $\square$

2. The inductive argument

We begin with a few basic results before we set up the proof. We proved above that if we have an Euler circuit then all vertices have even degree. And if there is an Euler path that is not a circuit, then there are exactly two vertices of odd degree. You might be wondering about the case when there is only one vertex of odd degree. This can't happen:

Proposition 5. *For any graph $G = (V, E)$, the size of the set $\{v \in V : d(v) \text{ is odd}\}$ is even.*

Proof. This follows directly from the handshake lemma, which states that

$$\sum_{v \in V} d(v) = 2|E|.$$

This tells us that the sum of the degrees of the vertices is always even. If adding a collection of integers produces an even number, then there must be an even number of odd numbers added. (The sum of any collection of even integers is even. If you add an even number of odd integers, you also get an even. If you add just one more odd number you'll get an odd result.) □

This means we don't have to worry about the case when there is only one vertex of odd degree!

In this section we will prove:

Theorem 6.

(1) *If G is a connected graph all of whose vertices have even degree, then G has an Euler circuit.*
(2) *If G is a connected graph where a and b are the only two vertices with odd degree, then G has an Euler path which starts at a and ends at b.*

Statement (1) here is exactly the harder implication from statement (1) of Theorem 1. Statement (2) differs slightly. First of all in Theorem 1, they say you get an Euler path when there are "at most two vertices." But because of Proposition 5, there can't be just one vertex with odd degree. If there are zero vertices of odd degree then by (1) there is an Euler circuit. But an Euler circuit is also an Euler path, so (1) already handles this case. The only other case is when there are exactly two vertices with odd degree, and this is what the new statement (2) handles.

We will prove this result using induction on the number of edges of the graph. So, we introduce three predicates defined for $n \geq 1$.

- $P(n)$: If G is a connected graph with n edges all of whose vertices have even degree, then G has an Euler circuit.
- $Q(n)$: If G is a connected graph with n edges where a and b are the only two vertices with odd degree, then G has an Euler path which starts at a and ends at b.
- $R(n)$: Both the statements $P(n)$ and $Q(n)$ are true.

In order to prove Theorem 6, we can prove that $R(n)$ is true for all $n \geq 1$. We will prove this by showing

- (Base case) $R(1)$ is true. That is, both $P(1)$ and $Q(1)$ are true.
- (Inductive step) For every $k \geq 1$, $R(k)$ implies $R(k+1)$. To show this, we get to assume that $P(k)$ and $Q(k)$ are true, and we need to prove $P(k+1)$ and $Q(k+1)$.

Remark 7. *There is also the case when there are no edges. But this case is somewhat silly because the only connected graph with no edges is the graph consisting of a single vertex. In this case the walk which visits just that single vertex (and doesn't move anywhere) is an Euler circuit. This is just to point out that the results here work in this silly case and we don't need to add a condition that the graph has at least on edge. We are leaving this case out of the induction because it might be confusing.*

Let's just get the base case out of the way.

Proof of the base case. We need to show $P(1)$ and $Q(1)$. These statements involve the case when our graph is connected and there is only one edge. In this case the graph must consist of exactly two vertices with one edge running between them. So, both vertices have degree one. Statement $P(1)$ is *vacuously true* in that there are no connected graphs with one edge all of whose vertices have even degree. We do need to check statement $Q(1)$. In this statement our two vertices are a and b and the only edge is $\{a, b\}$. The walk defined by $w_0 = a$ and $w_1 = b$ crosses our single edge exactly once and so is an Euler path. It also starts at a and ends at b so $Q(1)$ is true. □

The inductive step is harder. Again we get to assume that both $P(k)$ and $Q(k)$ are true, and we need to prove both $P(k+1)$ and $Q(k+1)$.

Let us first consider trying to prove $P(k+1)$. To this end, let G be a connected graph with $k+1$ edges all of whose vertices have even degree. Choose some vertex w_0 for our Euler cycle to start. Since our graph has edges and is connected, the vertex w_0 must be part of some edge say $\{w_0, w_1\}$. This determines the first two vertices of our walk. To make use of the inductive hypothesis, consider the subgraph G' obtained by removing the edge $\{w_0, w_1\}$. The subgraph G' has only k edges. The degree of w_0 and w_1 have gone down by one in G', and so w_0 and w_1 have odd degree in G'. The other vertices of G' all have even degree because we only removed this one edge. Setting $a = w_1$ and $b = w_0$, because $Q(k)$ is true, we know that if G' is connected, there is an Euler path for G' starting at w_1 and ending at w_0. This visits all edges of G' exactly once. So, the walk which starts by moving from w_0 to w_1 and then follows this Euler path in G' from w_1 back to w_0 gives an Euler circuit for G. This will prove that G has an Euler circuit as soon as we know that G' is connected. So, we get that $P(k+1)$ is true from the following result:

Lemma 8. *Suppose G is a connected graph all of whose vertices have even degree. Let $\{w_0, w_1\}$ be an edge of G and G' be the subgraph with just this edge removed. Then, G' is connected.*

In order to prove this Lemma, we first prove a condition that checks if the subgraph G' formed by removing a single edge is connected:

Proposition 9. *Let G be any connected graph and $\{w_0, w_1\}$ any edge. Let G' be the subgraph which consists of G with just the edge $\{w_0, w_1\}$ removed. Then G' is connected if and only if there is a walk in G' which starts at w_0 and ends at w_1.*

Proof. If there is no walk in G' from w_0 to w_1, then it is clear that G' is disconnected.

Now suppose that there is a walk from w_0 to w_1 in G'. Observe that because G is connected, for any two vertices a and b, there is a walk from a to b in G. This walk would also give a walk in G' unless it crosses the edge $\{w_0, w_1\}$. But if it does cross this edge, then we can follow the detour provided by our walk between w_0 and w_1 instead. We have shown that if there is a path from w_0 to w_1 in G' then we can move between any two vertices of G' by a walk in G'. So, G' is connected.

(*Remark:* Formally, the book says that a graph is connected if for any two vertices, there is a *path* between them. But if you can join two vertices by a walk in a graph then you can join them by a path in the same graph.) □

Proof of Lemma 8. We will give a proof by contradiction.

Assume to the contrary that G' is not connected. Then by Proposition 9, there is no walk from w_0 to w_1 in G'. Let A be the collection of vertices of G that can be reached by a walk from w_0 within G'. Similarly, let B be the collection of vertices of G that can be reached by a walk from w_1 within G'. Then A and B are disjoint sets of vertices (i.e., there are no vertices in common in these sets), because otherwise we'd have a walk from w_0 to w_1. Also, there can be no edge in G' joining a point in A to a point in B. Therefore every edge is either in E_A or E_B where E_A is the set of edges of G' joining points in A and E_B is the set of edges in G' joining points in B. Observe that $G'_0 = (A, E_A)$ and $G'_1 = (B, E_B)$ are graphs and every edge of G' appears in one graph or the other.

The only two vertices of G' with odd degree are w_0 and w_1 because we removed the edge between them. We have $w_0 \in A$ and $w_1 \in B$. So, the graphs G'_0 and G'_1 each have exactly one vertex of odd degree. But this is impossible because of Proposition 5! It follows that our assumption that G' was not connected must have been wrong. □

Definition 10 (Connected components)**.** The subgraphs $G'_0 = (A, E_A)$ and $G'_1 = (B, E_B)$ in the proof above are called the *connected components* of the disconnected graph G'. In general, if $G = (V, E)$ is a graph, then given any vertex a, we can consider the collection of all vertices $A \subseteq V$ which can be reached by a walk starting at a. The induced subgraph with vertex set A is the *connected component of* G *containing* a. (Recall that the *induced subgraph* is the subgraph with vertex set A where we also include all edges such that both endpoints lie in A.) If a graph G is connected, then its only connected component is the graph G itself. If G is not connected, then there are at least two different connected components of G. Every edge and every vertex is part of exactly one connected component. These ideas will show up again later.

It remains to use the truth of the statements $P(k)$ and $Q(k)$ to prove $Q(k+1)$. For the remainder of the section, we let G is a connected graph with $k+1$ edges where a and b are the only two vertices with odd degree. We need to show that G has an Euler path which starts at a and ends at b.

Our Euler path must start at a. The easiest case is when $d(a) = 1$, since it is clear how the Euler path must start:

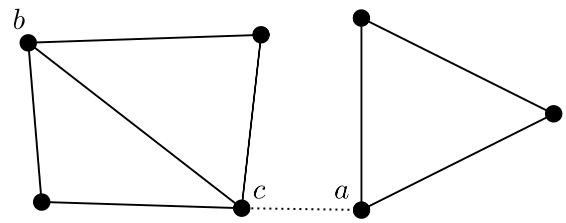
Lemma 11. *With hypotheses as above, if $d(a) = 1$, then the graph G has an Euler path.*

Proof. In this case, we can define a new graph G' by removing both a and the single edge leaving a. Clearly G' remains connected because the edge we removed led only to a which we also removed. The degrees of the vertices of G' are the same as those of G except that the degree of other endpoint of our edge has decreased by one. The graph G' has k edges, so we can hope to apply the truth of $P(k)$ and $Q(k)$.

If the edge we removed is $\{a, b\}$, then b now has even degree. Since G' is connected and all vertices have even degree, and since $P(k)$ is true, there is an Euler circuit on G' which we can take to start and end at b. An Euler path on G from a to b is then given by following the edge from a to b and then following the Euler circuit around G'.

Now suppose that the edge we removed is $\{a, a'\}$ for some $a' \neq b$. Then the graph G' is connected and has exactly two vertices of odd degree, namely a' and b. Since $Q(k)$ is true, we can find an Euler path on G' starting at a' and ending at b. Then the walk on G starting at a then moving to a' and then following the Euler path on G' from a' to b is an Euler path for G from a to b as required. □

Now let us begin to think about the case when $d(a) > 1$. Since the degree of a is odd, we must have at least three edges leaving a. We need to choose one to start our Euler path. In the proof of this case we will define G' to be G with just the edge removed and no vertices removed. The graph G' will be what we want to apply $P(k)$ or $Q(k)$ to as in the degree one case. But, unfortunately, we need be careful which edge we remove because if we remove the wrong edge it might disconnect the surface. For example, removing the dotted edge below leads to G' being disconnected:



But the solution to this is simply to choose a different edge leaving a to remove:

Lemma 12. *With hypotheses as above, if $d(a) \geq 3$, then there is at most one edge e leaving a such that removing e from G produces a disconnected subgraph.*

Proof. Suppose removing the edge $\{a, c\}$ from G disconnects the graph. We will show that if $\{a, d\}$ is a different edge that removing $\{a, d\}$ from G does not disconnect the surface.

Let G' be the graph obtained by removing $\{a, c\}$ from G. By Proposition 9, since G' is not connected, there is no walk in G' from a to c.

We claim that there is a walk in G from c to b that doesn't pass through a. This is clear if $b = c$. Now assume that $b \neq c$. In this case, the the only vertices in G' with odd degree are b and c. Therefore by Proposition 5, b and c must belong to the same connected component of G'. (If b and c belong to a different component, then each component has only one vertex with odd degree. But each component is a graph, and we can't have a graph with one odd degree vertex.) Therefore there is a walk from c to b in G'. This walk can not go through a because there is no walk from a to c. This proves our claim.

Now let G'' be the graph obtained by removing $\{a, d\}$ from G. We claim that G'' is connected. By Proposition 9, it is enough to check that there is a walk from a to d in G''. The previous paragraph gave us a walk from c to b in G that never passes through a (and so also doesn't move across edge $\{a, d\}$). So, there is a walk from a to b in G'' because we can first move across the edge $\{a, c\}$ and then follow this walk from c to b. To show G'' is connected, we can also find a walk from d to b, because then we can walk from a to d by first walking from a to b and then walking from b back to d. If $d = b$, then we'd be done. Now assume that $d \neq b$. Then the only two vertices of odd degree in G'' are b and d. Thus by Proposition 5, they must lie in the same connected component of G''. So, there is a walk from b to d. By earlier remarks, this gives us a walk from a to d and proves that G'' is connected. □

We now have all the pieces in place that we need to prove Theorem 6.

Proof of Theorem 6. We will prove that statement $R(n)$ are true for all n is true for all $n \geq 1$ by induction on n.

The base case of $n = 1$ was explained on page 4.

For the inductive step, let $k \geq 1$. We assume both $P(k)$ and $Q(k)$ are true. We need to prove that $P(k+1)$ and $Q(k+1)$ are true.

We will recall the proof of $P(k+1)$ here. Let G be a connected graph with $k+1$ edges all of whose vertices have even degree. Choose some vertex w_0 for our Euler cycle to start.

Since our graph has at least two edges and is connected, the vertex w_0 must be part of some edge say $\{w_0, w_1\}$. This determines the first two vertices of our walk. Let G' be the subgraph obtained by removing the edge $\{w_0, w_1\}$ from G. By Lemma 8, G' is connected. Since G has $k+1$ edges, G' has k edges. It also has exactly two vertices of odd degree, namely w_0 and w_1. Then since $Q(k)$ is true, there is a Euler path starting at w_1 and ending at w_0. The walk given by starting at w_0, then moving along $\{w_0, w_1\}$ to w_1, and then following the Euler path of G' from w_1 back to w_0 is an Euler cycle for G.

Now we will prove $Q(k+1)$. Let G be a connected graph with $k+1$ edges where a and b are the only two vertices with odd degree. We will find an Euler path from a to b.

If $d(a) = 1$, then Lemma 11 gives an Euler path from a to b.

Now assume that $d(a) \neq 1$. Since $d(a)$ is odd, we have $d(a) \geq 3$. Then by Lemma 12, we can find an edge $\{a, c\}$ such that the subgraph G' obtained by removing the edge $\{a, c\}$ from G is connected. Observe G' has k edges. We break into two cases.

First suppose that $c = b$. Then every vertex of G' has even degree (because both the degrees of a and b went down by one.) Since $P(k)$ is true, the subgraph G' has an Euler circuit, which we can take to start and end at b. So, an Euler path for G is given by starting at a, moving along edge $\{a, b\}$ to b and then following our Euler circuit of G'.

Now suppose that $c \neq b$. Then the subgraph G' has exactly two vertices of odd degree, c and b. Since G' is connected and has k vertices and since $Q(k)$ is true, there is an Euler path in G' from c to b. Then we get an Euler path for G by starting at a, moving along edge $\{a, c\}$ to c and then following our Euler path on G' from c to b.

By the Principle of Induction, the statement $R(n)$ is true for all $n \geq 1$. $\square$

References

[Lev21] Oscar Levin, *Discrete mathematics: An open introduction*, 3 ed., 2021, http://discrete.openmathbooks.org.

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