CHROMATIC INDEX OF COMPLETE GRAPHS

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These notes follow material in [Lev21, §4.4].

The *chromatic index* $\chi'(G)$ of a graph $G = (V, E)$ is the fewest number of colors that are possible in a proper edge coloring of G. (A *proper edge coloring* with n colors is a function $c : E \to \{1, 2, \ldots, n\}$ for which whenever two edges e_1 and e_2 share a vertex, we have $c(e_1) \neq c(e_2)$.)

The goal of this note is to compute the chromatic of the complete graph with n vertices, K_n. This answers a variant of the "Schoolgirl problem:"

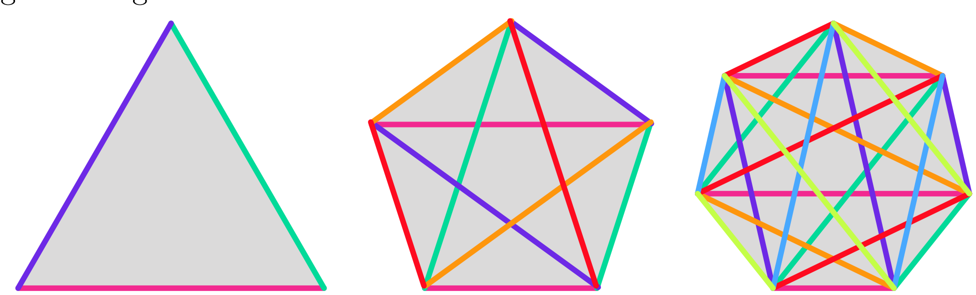
Question 1. Consider a class of n preschool students and a teacher. Whenever they cross a street, the students hold hands in pairs and any remaining student holds the teacher's hand. What are the fewest number of streets that can be crossed when every student has held every other student's hand at least once?

Here n is the number of students, and we can also consider these students to be the vertices of a complete graph K_n. We can assume our streets are colors (e.g., Green Street). So, whenever two students hold hands crossing a street, we color the edge between them the same color. The condition that every student must hold every other student's hand at least once, means that every edge gets colored at least once. Two edges sharing a vertex can't be colored the same color because it would mean that the corresponding student is holding two other people's hands. This is possible in real life, but we ruled it out by saying that the students hold their hands in pairs. So our coloring is proper. One issue is that in the problem two students could hold hands twice to cross different streets. This would result in an edge being colored twice (or more). But, we can deal with this by picking one of the colors used for each edge and not using the others to color the edge. We'd still get a proper coloring in the graph sense. One can see from this that the answer to the question is the chromatic index $\chi'(K_n)$.

See [Mal03, §4] for more about this problem.

Let's switch to thinking about $\chi'(K_n)$.

First consider the case when n is odd. Here we can arrange the vertices in the plane to be the vertices of a regular n-gon. We can color parallel edges the same colors producing a proper edge coloring as below.



Date: May 3, 2022.

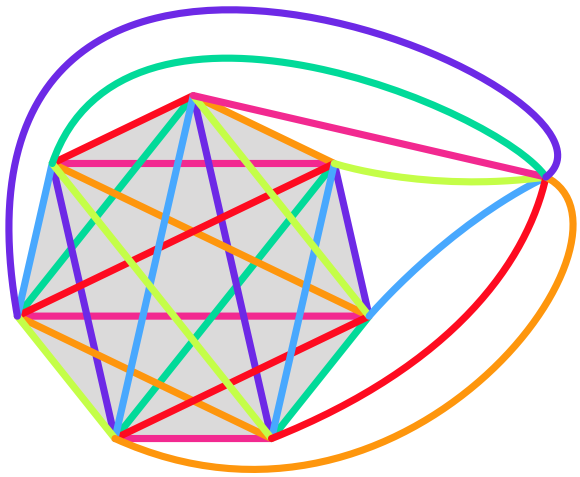
We have used n colors to give a proper coloring of K_n, so we have proved:

Proposition 2. *For $n \geq 3$ odd, we have $\chi'(K_n) \leq n$.*

This idea also gives some information about the chromatic index of complete graphs with an even number of edges.

Proposition 3. *For $n \geq 3$ odd, we have $\chi'(K_{n+1}) \leq n$.*

Proof. Let $n \geq 3$ be odd. Observe that in our coloring of K_n, each vertex does not have an edge of one of the colors we use. Construct K_{n+1} by adding a new vertex and joining it by an edge to every vertex. We color each edge from our new vertex to an old vertex the color that is missing for the edge from our old vertex. This gives a proper edge coloring of K_{n+1} with n colors. An example is shown below:



□

Recall that $\Delta(G)$ denotes the *maximal degree* of G, the largest degree of a vertex of G. In class we proved the following:

Proposition 4. *For any graph G we have $\chi'(G) \geq \Delta(G)$.*

Proof. We have a vertex of degree $\Delta(G)$, and we need at least $\Delta(G)$ colors to properly color all the edges leaving this vertex. □

We have $\Delta(K_n) = n - 1$, so as an application of Proposition 4 we get:

Corollary 5. *For any $n \geq 1$, $\chi'(K_n) \geq n - 1$.*

Combining Proposition 3 with Corollary 5, we see:

Theorem 6. *For any $n \geq 2$ even, we have $\chi'(K_n) = n - 1$.*

Proof. When $n \geq 4$, Proposition 3 tells us that $\chi'(K_n) \leq n - 1$. Corollary 5 tells us that $\chi'(K_n) \geq n - 1$. So, we must have $\chi'(K_n) = n - 1$.

The graph K_2 is just a single edge between two different vertices. So, no matter how we color the single edge, we'll do it with one color. Thus $\chi'(K_2) = 1$. □

When $k \geq 3$ is odd, Proposition 2 gives us that $\chi'(K_n) \leq n$ and Corollary 5 gives us that $\chi'(K_n) \geq n - 1$, so we know that $\chi'(K_n)$ is either $n - 1$ or n, but we're not sure which yet. We have:

Theorem 7. *When $n \geq 3$ is odd, we have $\chi'(K_n) = n$.*

Proof. Let $n \geq 3$ be odd. Suppose we have a proper coloring of K_n. Then we can have at most $\frac{n-1}{2}$ edges of each color. This is because our edges of the same color pair up vertices. Since n is odd, there must be at least one vertex left out. So, $n-1$ or fewer vertices are vertices of edges of this color.

The graph K_n has $\binom{n}{2} = \frac{n(n-1)}{2}$ edges. Since there are at most $\frac{n-1}{2}$ edges of each color and we have to color all edges, we need at least n colors. This proves that $\chi'(K_n) \geq n$.

We already know that $\chi'(K_n) \leq n$ by Proposition 2, so it must be that $\chi'(K_n) = n$. □

Combining our results we have that for all integers $n \geq 2$,

$$\chi'(K_n) = \begin{cases} n & \text{if } n \text{ is odd,} \\ n-1 & \text{if } n \text{ is even.} \end{cases}$$

References

[Lev21] Oscar Levin, *Discrete mathematics: An open introduction*, 3 ed., 2021, `http://discrete.openmathbooks.org`.

[Mal03] Joseph Malkevitch, *Colorful mathematics: Part IV*, Feature Column, American Mathematical Society, Dec 2003, `http://www.ams.org/publicoutreach/feature-column/fcarc-colorapp1`.

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