Math 36500: Combinatorics: Practice Final 2

In all parts of this problem consider only vertex colorings with three colors.

(a) The *path with n vertices* P_{n-1} is the graph whose vertex set is $\{1, 2, 3, \ldots, n\}$ such that for each $i \in \{1, \ldots, n-1\}$ there is an edge from i to $i+1$. How many different vertex colorings does P_{n-1} have? (Consider two vertex colorings to be different if any of the vertices are colored differently.)

(b) How many proper vertex colorings does P_{n-1} have?

(c) For $n \geq 1$, let a_n be the number of proper vertex colorings of P_{n-1} where vertices 1 and n have the same coloring (these are the first and last vertices). Let b_n be the number of vertex colorings where the vertices of 1 and n have different colorings. Work out the first three values of each of these sequences.

(d) For $n \geq 2$, find equations an equation for a_n in terms of a_{n-1} and b_{n-1} as well as for b_n in terms of a_{n-1} and b_{n-1}. (*Hint:* If you have a proper vertex coloring of a path and you delete the last vertex and edge, what happens?)

(e) Combine the equations found in part (d) to give a formula for b_n in terms of b_{n-1} and b_{n-2} for $n \geq 3$. This formula should be a standard recurrence relation and together with initial conditions from (c) gives a recursive definition for the sequence $\{b_n\}_{n\geq 1}$. Solve this recursive definition to find a closed formula for b_n.

(f) Explain why b_n is also the number of proper colorings of a cycle with n vertices.