The Logistic Family

We had a worksheet in class that did several things:

1. Explained the dynamics of quadratic maps with zero or one fixed point.
2. Proved that if a quadratic has two fixed points, it is conjugate via an affine linear map to a map of the form
$$F_\mu(x) = \mu x(1-x).$$
where $\mu > 1$. These maps form the *Logistic family of maps*.

One trivial remark is that if $\mu > 1$, then every point in $(-\infty, 0) \cup (1, \infty)$ is forward asymptotic to $-\infty$. This is because $F_\mu(x) < x$ whenever $x \in (-\infty, 0)$ guaranteeing from prior arguments that points in $(-\infty, 0)$ tend to $-\infty$. Also if $x > 1$, then $F_\mu(x) < 0$, so again x will be forward asymptotic to $-\infty$.

Because of this we will concentrate on understanding the dynamics on the interval $[0, 1]$.

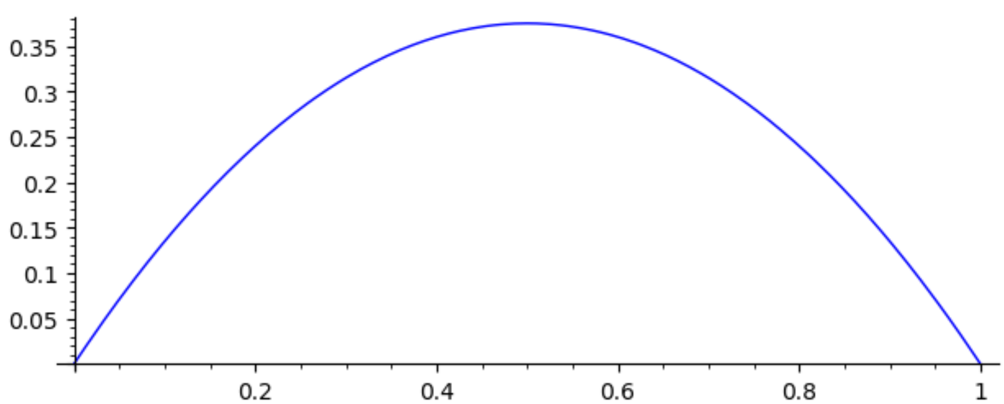
The goal of this notebook is to take a tour through the Logistic family as μ increases from the value one. Below we define the logistic family:

In [1]:
```
def F(mu):
    def F_mu(x):
        return mu*x*(1-x)
    return F_mu
```

For example F(3/2) can be plotted as follows:

In [2]:
```
G = F(3/2)
plot(G, 0, 1, aspect_ratio = 1)
```

Out[2]:



The value μ represents $F'(0)$. Observe also that zero is fixed. Since $\mu > 0$, this point represents a repelling fixed point.

The other fixed point is at the point

$$p_\mu = \frac{\mu - 1}{\mu}.$$

We define this point as a function of mu:

In [3]:
```
def p(mu):
    return (mu-1)/mu
```

We can check symbolically that p_μ is indeed fixed by F_μ:

In [4]:
```
mu = var("mu") # make mu into a symbolic variable
bool(F(mu)(p(mu)) == p(mu)) # Attempt to evaluate the equation as true or fal
```

Out[4]:
```
True
```

More on symbolic expressions can be found here: [http://doc.sagemath.org/html/en/reference/calculus/sage/symbolic/expression.html (http://doc.sagemath.org/html/en/reference/calculus/sage/symbolic/expression.html)](http://doc.sagemath.org/html/en/reference/calculus/sage/symbolic/expression.html)

Now we consider the multiplier of the fixed point p_μ. This is just the value $F_\mu'(p_\mu)$. Here we have Sage compute F_μ':

In [5]:
```
x = var("x")
F_prime = F(mu)(x).derivative(x)
F_prime
```

Out[5]:
```
-mu*(x - 1) - mu*x
```

Below we demonstrate that

$$F_\mu'(p_\mu) = 2 - \mu.$$

Note that `F_prime` is an algebraic expression in the variables `x` and `mu`. We can substitute a value for `x` using the `subs()` method which takes as input a mapping. We will map `x` to `p(mu)`. The `.simplify_full()` method attempts to simplify the resulting expression.

In [6]:
```
F_prime.subs({x:p(mu)}).simplify_full()
```

Out[6]:
```
-mu + 2
```

Observe that:

1. We have $0 < F'_\mu(p_\mu) < 1$ when $\mu \in (1,2)$. This means that p_μ is an attracting fixed point, and that F'_μ is a one-to-one orientation preserving map in a sufficiently small open neighborhood of p_μ. (An {\em open neighborhood of p_μ is an open set containing p_μ such as the interval $(p_\mu - \epsilon, p_\mu + \epsilon)$ for $\epsilon > 0$ small.)
2. In the case $\mu = 2$, we have $F'_\mu(p_\mu) = 0$. This means that $p_\mu = \frac{1}{2}$ since $\frac{1}{2}$ is the only critical point. Since $F'_\mu(p_\mu) = 0$, p_μ is a {\em super-attracting fixed point}. Furthemore, because p_μ coincides with the critical point, the map F_μ is never one-to-one on a neighborhood of p_μ.
3. We have $-1 < F'_\mu(p_\mu) < 0$ when $\mu \in (2,3)$. This means that p_μ is an attracting fixed point, that F'_μ is a one-to-one orientation-reversing map in a small neighborhood of p_μ.
4. When $\mu > 3$, we have that $F'_\mu(p_\mu) < -1$. At this point p_μ has become a repelling fixed point.

It follows from the above facts that two maps taken from different cases above are not topologically conjugate. For example, a map from case 1 is not conjugate to a map from case 3, because in case 1, the attracting fixed point p_μ is locally orientation preserving, whiel in case 3 the attracting fixed point is locally orienation reversing.

In fact it can be shown that two maps taken from case 1 are topologically conjugate, and two maps taken from case 3 are topologically conjugate. The topological conjugacy can not be a diffeomorphism because conjugacy by a diffeomorphism preserves multipliers at fixed and periodic points. (Excercise: Show this is true.)

Now we will attempt to understand the dynamics of these maps for values of μ running from 1 to a little bigger than 3.

The case when $\mu \in (1,2)$.

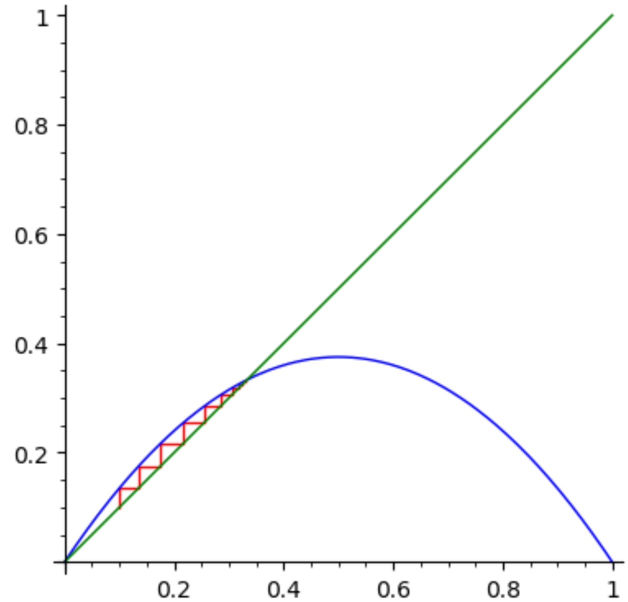
This cobweb function was taken from an earlier notebook:

In [7]:
```
def cobweb(x, T, N, xmin, xmax):
    cobweb_path = [(x,x)]
    for i in range(N):
        y = T(x) # Reassign y to be T(x).
        cobweb_path.append( (x,y) )
        cobweb_path.append( (y,y) )
        x = y # Reassign x to be identical to y.
    cobweb_plot = line2d(cobweb_path, color="red", aspect_ratio=1)

    function_graph = plot(T, (xmin, xmax), color="blue")

    # define the identity map:
    identity(t) = t
    id_graph = plot(identity, (xmin, xmax), color="green")

    return cobweb_plot + function_graph + id_graph
```

Here is an example of a cobweb plot in the case $\mu = \frac{3}{2}$ starting at $x = 0.1$, plotting 10 iterations over the interval $(0, 1)$.

In [8]:
```
cobweb(0.1, F(3/2), 10, 0, 1)
```

Out[8]:



We will use sliders to allow experimentation. A slider can be created decribing values in the interval $[1, 2]$ with a step size of 0.001 and initial value $3/2$ as below:

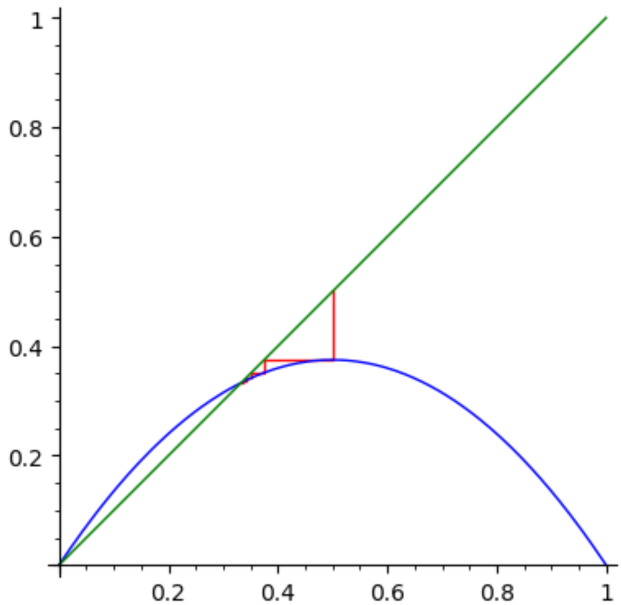
In [9]:
```
slider(1, 2, 0.001, 3/2)
```

We want to be able to vary $\mu \in (1, 2)$ and vary $x \in (0, 1)$. We can use the `@interact` decorator for a function to do this. The values of the sliders will be used as input to a function which is run whenever the sliders are updated.

```
In [10]: @interact
def iteractive_plot(mu = slider(1, 2, 0.001, 3/2),
                    x = slider(0, 1, 0.001, 1/2)):
    return cobweb(x, F(mu), 10, 0, 1)
```

mu 1.50

x 0.50



From looking at the Cobweb plot, you should be convinced that:

1. Any point $x \in (0, p_\mu)$ has an orbit which increases and accumulates on p_μ. To prove this, it suffices to show that $x \in (0, p_\mu)$ implies $x < F_\mu(x) < p_\mu$ and apply our standard argument.
2. Any point $x \in (p_\mu, \frac{1}{2}]$ has an orbit which decreases down toward p_μ. Again it suffices to show that if $x \in (p_\mu, \frac{1}{2}]$, then $p_\mu < F_\mu(x) < x$.
3. If $x \in (\frac{1}{2}, 1)$, then $0 < F_\mu(x) < \frac{1}{2}$. From this and statements 1 and 2 above, it follows that x is forward asymptotic to p_μ.

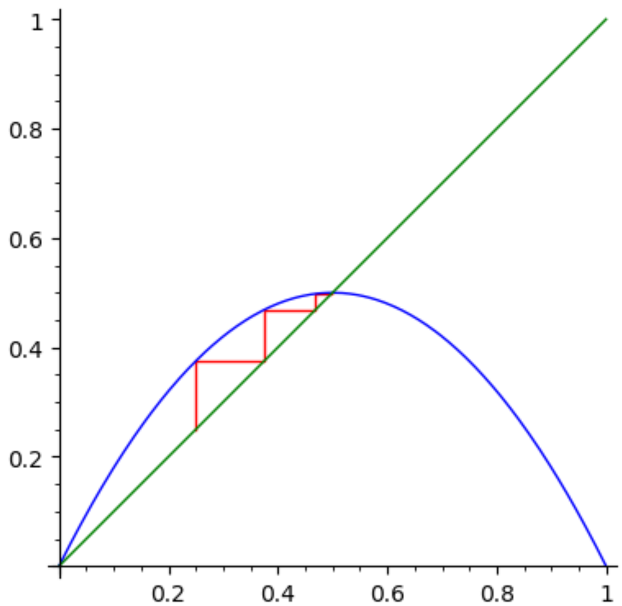
The above shows that $W^s(p_\mu) = (0, 1)$, which completely describes the dynamics on $[0, 1]$. Every point in $(0, 1)$ is forward asymptotic to p_μ. (Also, zero is fixed and $F_\mu(1) = 0$.)

The case $\mu = 2$.

Recall that F_2 has a super-attracting fixed point. The point $p_\mu = \frac{1}{2}$ is both a critical point and fixed. The following code lets you experiment with this case.

```
In [11]: @interact
def iteractive_plot(x = slider(0, 1, 0.001, 1/4)):
    return cobweb(x, F(2), 10, 0, 1)
```

x 0.25

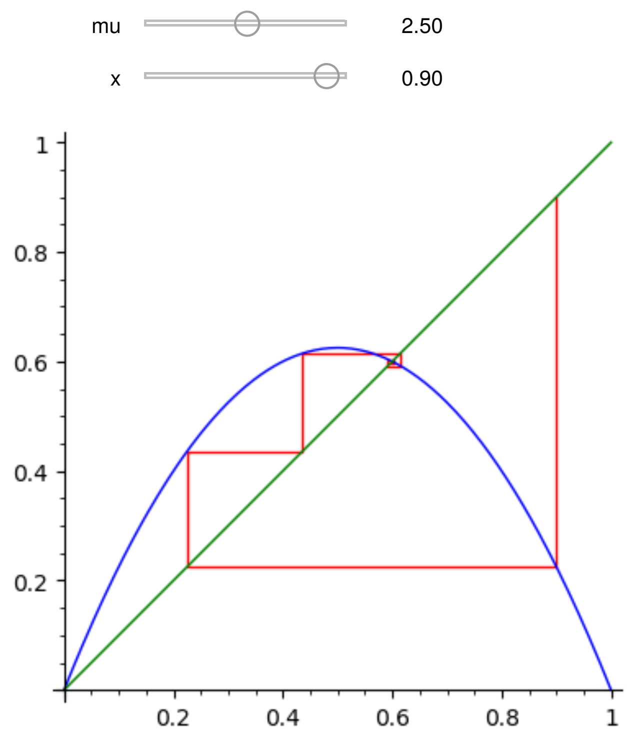


Similar analysis to the previous case can be used to prove that every point in $(0, 1)$ is forward asymptotic to the super-attracting fixed point $p_\mu = \frac{1}{2}$.

The case of $\mu \in (2, 3)$.

You can experiment with the maps below:

In [12]:
```
@interact
def iteractive_plot(mu = slider(2, 3, 0.001, 2.5),
                    x = slider(0, 1, 0.001, 0.9)):
    return cobweb(x, F(mu), 20, 0, 1)
```



The dynamics are a bit more complex because locally F_μ is orientatation-reversing in a neighborhood of p_μ. This causes orbits to spiral inward rather than approach directly.

By experimenting with the cobweb plots above, you should be convinced that all orbits are asymptotic to the fixed point p_μ.

Theorem. When $2 < \mu < 3$, all orbits in $(0, 1)$ are asymptotic to p_μ.

We will give a proof of this using the following claim about the interval $I = [\frac{1}{2}, 2p_\mu - \frac{1}{2}]$.

Claim. Suppose $2 < \mu < 3$.

1. The interval is symmetric around p_μ.
2. We have $F_\mu(\frac{1}{2}) \in I$. Note that $F_\mu(\frac{1}{2})$ is the maximum value taken by F_μ.
3. We have $-1 < (F_\mu^2)'(x) < 1$ for each $x \in I$.

Proof of 1. It is symmetric around p_μ because the endpoints are at equal distance from p_μ. Observe

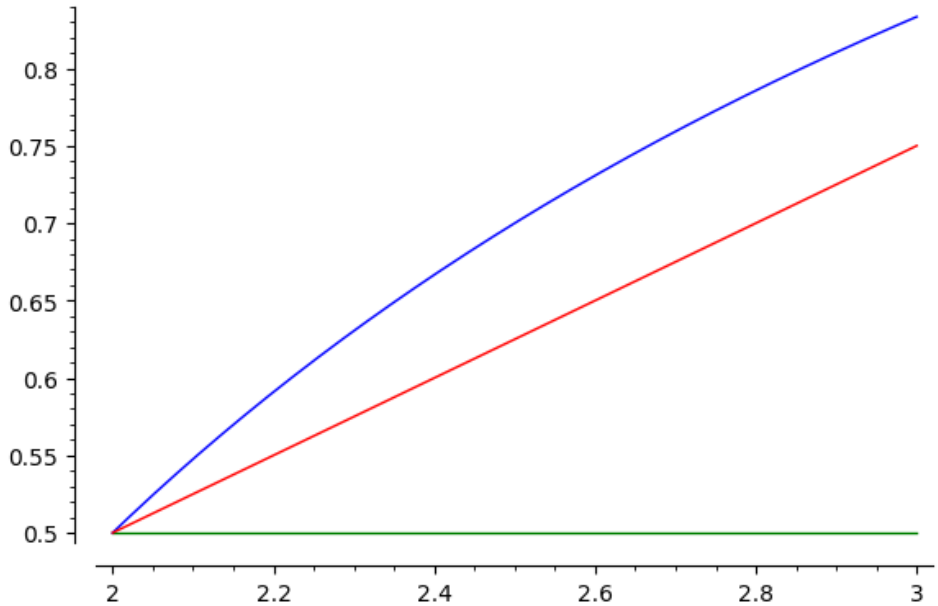
$$|p_\mu - \frac{1}{2}| = p_\mu - \frac{1}{2} = |p_\mu - (2p_\mu - \frac{1}{2})|.$$

Graphical "proof" of 2. We can consider plotting the left and right endpoints of I as well as $F_\mu(\frac{1}{2})$. We plot the left endpoint in green, the right endpoint in blue, and the $F_\mu(\frac{1}{2})$ in red below. All are expressed as a function of μ.

In [13]:

```
plt = plot(1/2, 2, 3, color="green")
plt += plot(2*p(mu)-1/2, 2, 3, color="blue")
plt += plot(F(mu)(1/2), 2, 3, color="red")
plt
```

Out[13]:

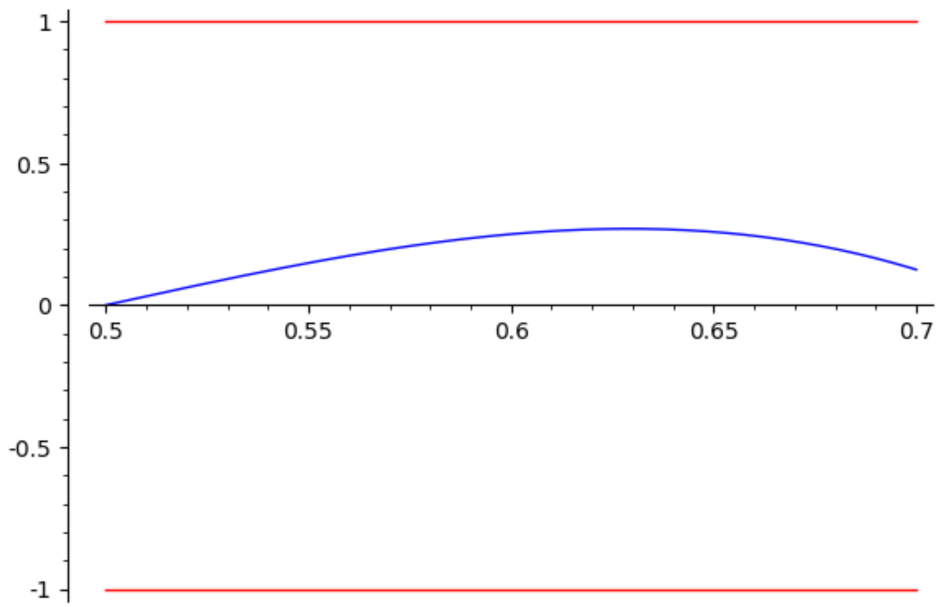


Graphical "proof" of 3. We plot $(F_\mu^2)'(x)$ as a function of x below, allowing the choice of μ with a slider. We also add plots of the constant function -1 and the constant function 1.

In [14]:

```
@interact
def iteractive_plot(mu = slider(2, 3, 0.001, 2.5)):
    x = var("x")
    F_mu = F(mu)
    square = F_mu(F_mu(x))
    plt = plot(square.derivative(x), 1/2, 2*p(mu)-1/2, color="blue")
    plt += plot(-1, 1/2, 2*p(mu)-1/2, color="red")
    plt += plot(1, 1/2, 2*p(mu)-1/2, color="red")
    return plt
```

mu 2.50



Proposition. If $x \in I$, then the orbit of x is forward asymptotic to p_μ.

Proof: We use the Claim. Since $|(F_\mu^2)'(t)|$ is a continuous function of t, it attains a maximum on I. Call this value C. By statement 3 of the claim, we know $C < 1$. Then by the Mean Value Theorem, we see that for any $x \in I$, we have

$$|F_\mu^2(x) - p_\mu| < C|x - p_\mu|.$$

Since $F_\mu^2(x)$ is closer to p_μ than x and I is symmetric around p_μ, it must be that $F_\mu^2(x) \in I$. Then by induction we see that for any $x \in I$ and any $k > 0$, we have

$$|F_\mu^{2k}(x) - p_\mu| < C^k|x - p_\mu|.$$

Since $C < 1$, the right hand side tends to zero as $k \to +\infty$. Thus, we have that $\lim_{k\to+\infty} F_\mu^{2k}(x) = p_\mu$. This shows that the orbit of x is forward asymptotic to p_μ. as desired. □

Proof of the Theorem. Now we will show that all points are forward asymptotic to p_μ.

From the proposition above, we already know that the statement is true on the interval $I = [\frac{1}{2}, 2p_\mu - \frac{1}{2}]$.

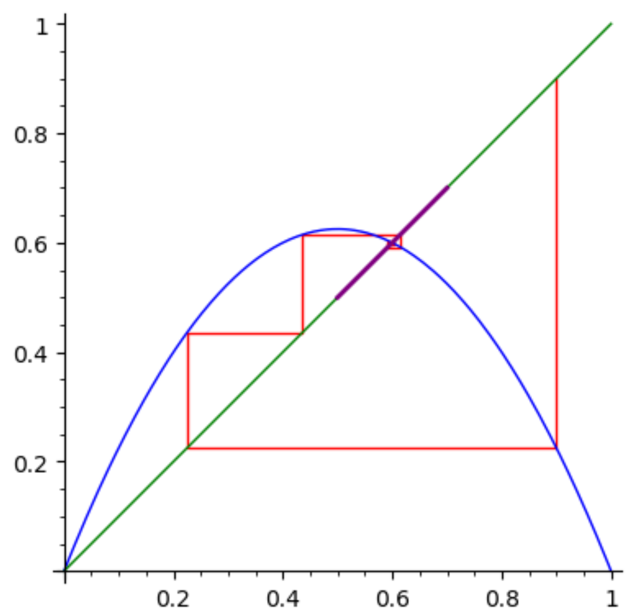
Now consider the case of $x \in (0, \frac{1}{2})$. Observe that if $x \in (0, \frac{1}{2})$, then $F_\mu(x) > x$. Since there are no fixed points in the interval $(0, \frac{1}{2})$, points in the orbit increase until at some point we reach a $F_\mu^n(x) \geq \frac{1}{2}$. Since $F_\mu^n(x)$ is in the image of F_μ, it is less than or equal to the maximum $F_\mu(\frac{1}{2})$ taken. Thus from statement (2) of the claim we know that $F_\mu^n(x) \in I$. But then it follows from the Proposition above that $F_\mu^n(x)$ is forward asymptotic to p_μ. But, then x must be forward asymptotic to p_μ as well.

We already know $\frac{1}{2}$ is forward asymptotic to p_μ since $\frac{1}{2} \in I$. Now consider the $x > \frac{1}{2}$. Let $y = 1 - x$, which is less than $\frac{1}{2}$. Then we know from the previous paragraph that y is forward asymptotic to p_μ. But we also have that $F_\mu(x) = F_\mu(y)$ and thus $F_\mu^n(x) = F_\mu^n(y)$ for all $n \geq 1$. Thus x must also be forward asymptotic to p_μ. □

In [15]:
```
@interact
def iteractive_plot(mu = slider(2, 3, 0.001, 2.5),
                    x = slider(0, 1, 0.001, 0.9)):
    plt = cobweb(x, F(mu), 20, 0, 1)
    plt += line2d([(1/2,1/2),(2*p(mu)-1/2,2*p(mu)-1/2)],thickness=2,color="pu
    return plt
```

mu 2.50

x 0.90

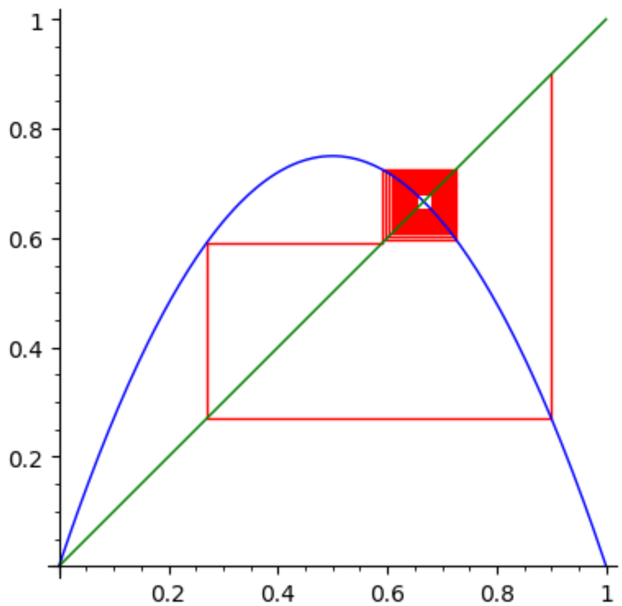


Passing through $\mu = 3$.

At the value of 3, the point p_μ is slowly attracting.

In [16]:
```
@interact
def iteractive_plot(x = slider(0, 1, 0.001, 0.9)):
    return cobweb(x, F(3), 200, 0, 1)
```

x 0.90

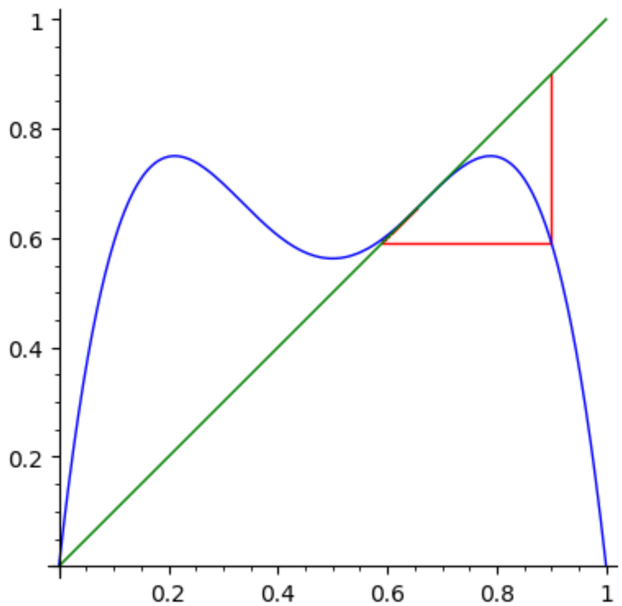


Aside from looking at the cobweb plot above, a good way to convince yourself of this is to look at the square. Here we define the square $F_\mu^2(x)$:

In [17]:
```
def F2(mu):
    F_mu = F(mu)
    def F2_mu(x):
        return F_mu(F_mu(x))
    return F2_mu
```

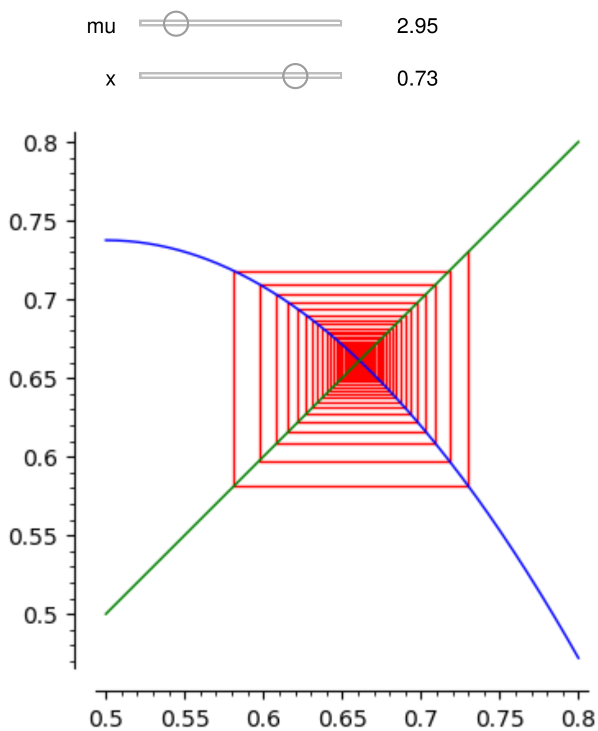
In [18]:
```
@interact
def iteractive_plot(x = slider(0, 1, 0.001, 0.9)):
    return cobweb(x, F2(3), 200, 0, 1)
```

x 0.90



The following lets you see what happens when you vary μ through the value of 3. We plot on a small interval containing p_μ.

In [19]:
```
@interact
def iteractive_plot(mu = slider(2.9, 3.2, 0.001, 2.95),
                    x = slider(0.5, 0.8, 0.001, 0.73)):
    return cobweb(x, F(mu), 200, 0.5, 0.8)
```



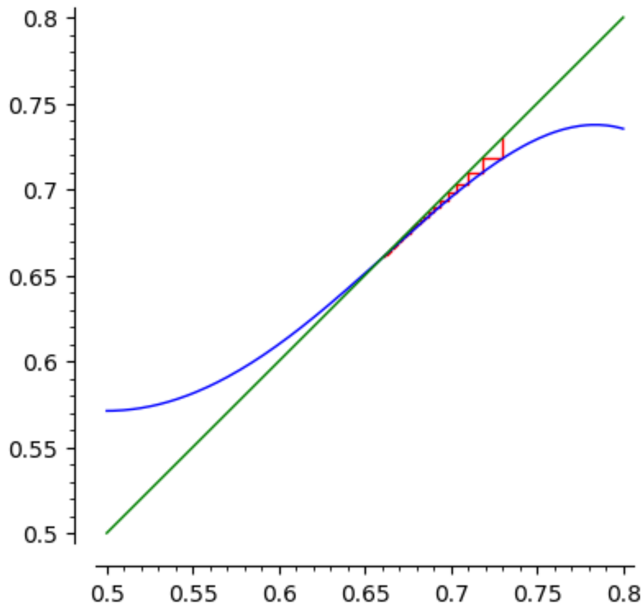
It is easier to see what is going on by plotting F_μ^2.

In [20]:

```
@interact
def iteractive_plot(mu = slider(2.9, 3.2, 0.001, 2.95),
                    x = slider(0.5, 0.8, 0.001, 0.73)):
    return cobweb(x, F2(mu), 100, 0.5, 0.8)
```

mu 2.95

x 0.73



The family of maps F_μ undergoes a period-doubling bifurcation at the value $c = 3$. At values of c slightly greater than 3, the fixed point p_μ has switched to being a repelling fixed point, and a new attracting period two orbit emerges.