NOTE ON STABLE SET DEFINITIONS

W. PATRICK HOOPER

In section I.3, Devaney gives two definitions of forward asymptotic:

1. If p is a periodic point of period n for f, he says: "A point x is *forward asymptotic* to p if $\lim_{i\to\infty} f^{in}(x) = p$."
2. If p is non-periodic, he indicates "A point x is forward asymptotic to p if $\lim_{i\to\infty} |f^i(x) - f^i(p)| = 0$."

Of course, we could use definition 2 even if p is periodic. The following is implicit but not explicit in Devaney:

Proposition 1. *Suppose f is continuous. If p is a periodic point for f, x is forward asymptotic to p in the sense of definition 1 if and only if is forward asymptotic in the sense of definition 2.*

Proof. That definition 2 implies 1 should be clear. If $\lim_{i\to\infty} |f^i(x) - f^i(p)| = 0$, then we also have $\lim_{i\to\infty} |f^{in}(x) - f^i(pn)| = 0$. because we are passing to a subsequence. But since p is period n, we get that $\lim_{i\to\infty} |f^{in}(x) - p| = 0$, and this implies that $\lim_{i\to\infty} f^{in}(x) = p$.

Now suppose that definition 1 is satisfied, i.e., $\lim_{i\to\infty} f^{in}(x) = p$. As in definition 2, define the sequence

$$s_k = |f^k(x) - f^k(p)|.$$

We need to show that $s_k \to 0$ as $k \to \infty$.

Define $p_j = f^j(p)$ for $j = 0, \ldots, n-1$. Then $\{p_0, p_1, p_2, \ldots, p_{n-1}\}$ is the periodic orbit of p, and all these points are period n. Observe that by continuity of f^j, we have that

$$\lim_{i\to\infty} f^{in+j}(x) = f^j(p) = p_j \quad \text{for all } j \in \{0, 1, \ldots, n-1\}.$$

Noting that $p_j = f^j(p) = f^j \circ f^{in}(p) = f^{in+j}(p)$ because p is period n, we see that

$$\lim_{i\to\infty} |f^{in+j}(x) - f^{in+j}(p)| = 0 \quad \text{for all } j \in \{0, 1, \ldots, n-1\}.$$

So, we've shown that

$$\lim_{i\to\infty} s_{in+j} = 0 \quad \text{for all } j \in \{0, 1, \ldots, n-1\}.$$

What we've just done is partition the sequence $\{s_k\}$ into subsequences consisting of indices that are equivalent modulo n. It is a basic observation that because all indices are included and the subsequences all converge to zero, then the sequence s_k must also converge to zero. To see why observe that for all $\epsilon > 0$ there are constants I_j such that

$$i > I_j \quad \text{implies} \quad |s_{in+j}| < \epsilon.$$

Set $K = n\big(\max\{I_j : j = 0, \ldots n-1\} + 1\big)$. Then $k > K$ implies that $k = in + j$ for some $j \in \{0, \ldots, n-1\}$ and $i > I_j$. We therefore have that

$$|s_k| = |s_{in+j}| < \epsilon$$

proving that $s_k \to 0$. ∎

The City College of New York, New York, NY, USA 10031
Email address: whooper@ccny.cuny.edu

Date: August 31, 2022.