FINAL PROJECTS FOR MATH A4500: INTRODUCTION TO DYNAMICAL SYSTEMS

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In this course, we will go through many examples of dynamical systems, with the primary goal being to introduce the tools used to understand these systems.

I hope that the final projects will increase the classes perspective of the breadth of the subject of dynamical systems and connections to other areas of mathematics.

1. Directions

You are expected to give a presentation on a topic in Dynamical Systems which we have not covered in the course. Presentations will be given at the time our Final exam is scheduled (6:00pm - 8:15pm on Tuesday, December 20th) and should be 10-20 minutes long.

Projects will be carried out by individuals. If multiple people are interested in the same topic, we can discuss how to divide the topic into pieces so that multiple different presentations can be made.

You must make some slides for the material you present. The slides will be part of the grade and must be submitted for grading. As such, slides should include all topics you present.

Slides can be created with latex or any other presentation software. If possible, please include some pictures of the system (ideally which you have created). Include a slide of citations and inline citations on the slides. (Be mindful of not committing plagiarism.)

I recommend that you come to office hours or schedule a meeting with me sometime before the presentation to discuss your progress.

2. Possible projects

I list some ideas for possible projects below. If you have your own interests you may use them to design a project as well. **Please email me your project topic once you have decided on it.** I would like to keep track of who is working on which project so that I can coordinate things as necessary.

It should not be too hard to find references on these topics. It is okay to use wikipedia and similar websites to get started, but *your references should include academic sources (e.g., research papers, research books, and textbooks on dynamics)*. Contact me if you are having trouble and need suggestions.

2.1. Iterated function systems.

Iterated function systems can be used to construct many famous and natural fractals. An example is shown in Figure 1.

Possible projects:

- Define an iterated function system as a dynamical system. Describe how fractals arise from this construction. In what sense are they self-similar?

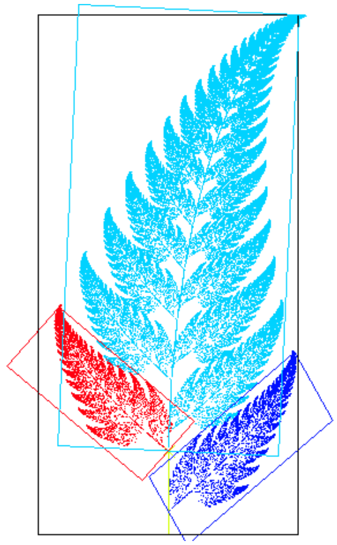

FIGURE 1. The limit set of an iterated function system. This graphic is in the public domain.

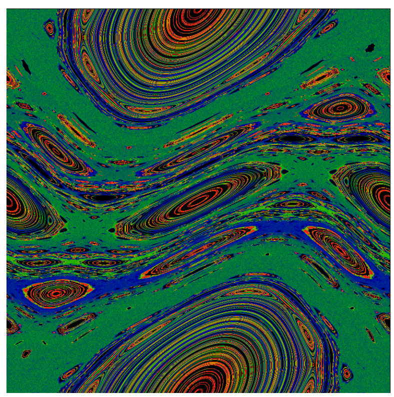

FIGURE 2. Example orbit plot of the standard map. Image was created by Linas and released under a Creative Commons Attribution-Share Alike 3.0 Unported license. See `https://commons.wikimedia.org/wiki/File:Std-map-0.971635.png`

- Discuss the definition of a fractal dimension (e.g., box dimension or Hausdorff dimension). Describe the fractal dimension of some sets obtained by iterated function systems.

2.2. **The standard map.** The standard map is really a family of area preserving maps of the torus $\mathbb{T}^2$. Dynamics for many parameters are characterized regions where behavior is quasi-periodic in some regions consisting of periodic islands surrounding periodic islands, and in other regions (apparent) chaotic behavior. See Figure 2. This is a principal example in Kolmogorov-Arnold-Moser Theory (KAM theory).

Possible projects:

- Define the standard map and draw some pictures of orbits as the parameter varies.

2.3. **Irrational rotations.** Irrational rotations have dense orbits. In fact more is true. Let $R : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be an irrational rotation. If $I = (a, b)$ is an interval in $\mathbb{R}/\mathbb{Z}$ with $0 \leq b - a \leq 1$, then for any $x_0 \in \mathbb{R}/\mathbb{Z}$,

$$\lim_{N\to\infty} \frac{1}{N} \#\{n \in \mathbb{N} : \ 0 \leq n < N \text{ and } R^n(x_0) \in I\} = b - a$$

So, in some sense orbits spend the "right" amount of time in each interval. This is a consequence of the fact that the system is *uniquely ergodic* and can be seen by an application of the Birkhoff Ergodic Theorem.

It would probably be best to have a bit of background in graduate real analysis so that you know what a measure is.

Possible projects:

- Explain what ergodicity and unique ergodicity means. Describe how unique ergodicity gives the limiting result above through the use of the Birkhoff Ergodic Theorem. How does unique ergodicity make this result stronger?

2.4. **Odometers.** Let Σ_2 denote the set of all 1-sided sequences in the alphabets $\mathcal{A} = \{0, 1\}$. Recall this means that an element of Σ_2 has the form

$$s = (s_0 s_1 s_2 \ldots) \quad \text{where each } s_i \in \mathcal{A}.$$

The 2-*adic odometer* is the map $f : \Sigma_2 \to \Sigma_2$ which adds one with carry to the right. So for example

$$f(1\ 1\ 1\ 0\ 1\ 0\ 1\ 1\ 0 \ldots) = (0\ 0\ 0\ 1\ 1\ 0\ 1\ 1\ 0 \ldots).$$

Now we will give a formal definition. Given $s \in \Sigma_2$, let $\nu(s) \in \mathbb{N} \cup \{+\infty\}$ be the minimal index i so that $s_i = 0$ or $+\infty$ if no such i exists. Then we define $f(s) \in \Sigma_2$ by the rule:

$$f(s)_i = \begin{cases} 0 & \text{if } i < \nu(s), \\ 1 & \text{if } i = \nu(s), \\ s_i & \text{if } i > \nu(s). \end{cases}$$

Possible topics:

- Discuss why *all* orbits are dense. (A *minimal system* is a dynamical system in which all orbits are dense.)
- The map is *renormalizable*. Discuss what that means in this case and what it can say about the dynamics in terms of the visits of orbits to cylinder sets.
- The map is uniquely ergodic. What does that say about orbits?

2.5. **Denjoy Construction.** The Denjoy construction produces orientation preserving homeomorphisms of the circle which are not conjugate to rotations.

Possible projects:

- Describe Denjoy's construction. How do circle homeomorphisms constructed in this way differ from rotations of the circle? Describe simple maps of the circle which give rise to such non-standard circle maps.

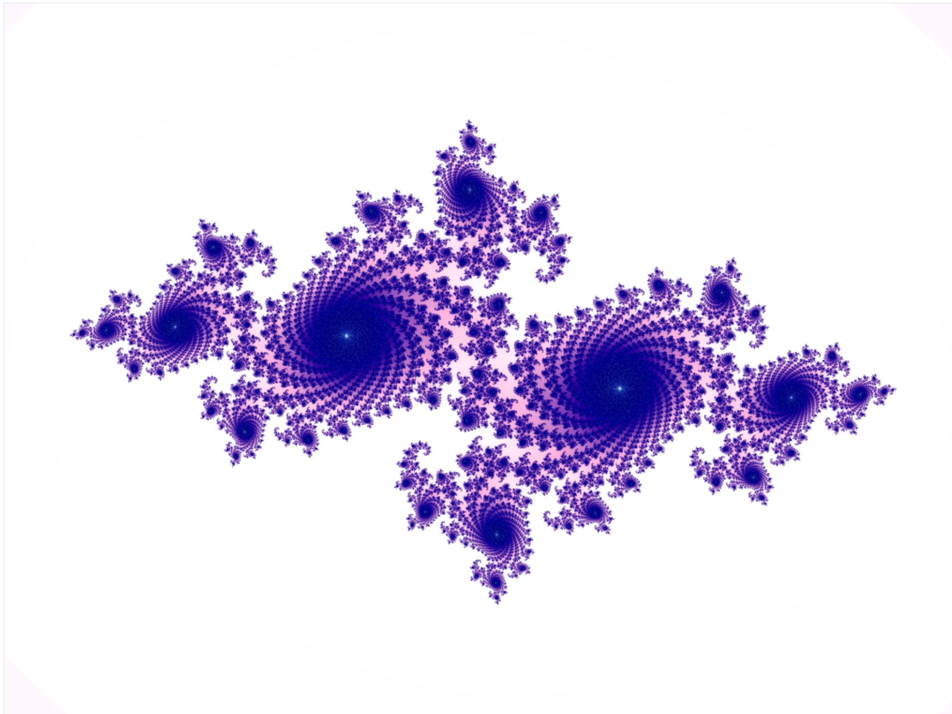

FIGURE 3. Example Julia set. This figure is in the public domain.

2.6. **Complex Dynamics.** Complex dynamics is a beautiful topic we have not discussed.

If $f : \mathbb{C} \to \mathbb{C}$ is a complex polynomial, e.g., $f(z) = z^3 - 2z + i$, the *filled Julia set* is the set of z so that $\lim f^n(z) \neq \infty$. These sets are often have fractal boundary. See Figure 3.

Possible projects:

- Define Julia set and give some examples. Describe how pictures of Julia sets are drawn (by computer).
- The Mandelbrot set is the set of $c \in \mathbb{C}$ so that $\lim f_c^n(0) \neq \infty$ where $f_c(z) = z^2 + c$. Describe how the filled Julia set of f_c changes depending if c lies in the Mandelbrot set or not. Why is 0 so important here? (Note that the maps f_c for $c \in \mathbb{R}$ are affinely conjugate to maps in the Logistic family, though defined on $\mathbb{C}$.)

2.7. **Rotation number.** The rotation number is an element of $\mathbb{R}/\mathbb{Z}$ associated to an orientation preserving homeomorphism of the circle.

Possible projects:

- What is the definition of rotation number? In what circumstances can you say that an orientation preserving homeomorphism is topologically conjugate to a rotation? How about semi-conjugate?

2.8. **The Perron-Frobenius theorem.** Suppose that M is an $n \times n$ matrix with positive entries. Let U be the set of unit vectors in $\mathbb{R}^n$ all of whose entries are non-negative. Consider the map

$$f : U \to U; \quad f(\mathbf{v}) = \frac{1}{\|M\mathbf{v}\|} M\mathbf{v},$$

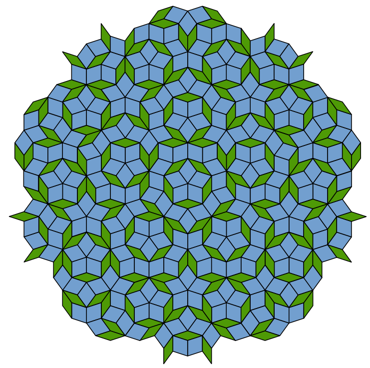

FIGURE 4. A subset of a Penrose tiling. This figure is in the public domain.

where $\| \cdot \|$ denotes the length of a vector. (The scalar $\frac{1}{\|M\mathbf{v}\|}$ is there to scale $M\mathbf{v}$ to unit length.)

The Perron-Frobenius theorem says that there is a unique $\mathbf{v}_0 \in U$ so that $M(\mathbf{v}_0) = \mathbf{v}_0$ and furthermore $\lim_{n\to\infty} f^n(\mathbf{v}) = \mathbf{v}_0$ for any $\mathbf{v} \in U$. (In fact the Perron-Frobenius theorem is slightly more general.)

Possible projects:

- State and give a sketch of the proof of the Perron-Frobenius theorem.
- Explain the connection to Google's PageRank algorithm. See e.g., [BL06].

2.9. **Information theory.** Information theory concerns the storing of information as a string of characters (perhaps zeros and ones as in a computer). *Information entropy* is a measure of how much information can be stored in such a sequence.

Possible projects:

- Consider the information content of strings coming from a full shift space and a shift of finite type. What is the information entropy in this case and what does this quantity mean?

2.10. **Tiling spaces.** A tiling is a partition of the plane into tiles which usually only have a fixed set of shapes (up to translation). These shapes are called *tiles*. See Figure 4 for an example. Sometimes there are additional rules restricting how tiles can meet. The tiling is periodic if there is a translation which preserves the tiling and aperiodic if no such translation exists.

Often we consider a finite list of tiles and consider all tilings that can be made from those tiles. The collection of all such tilings is a *tiling space*. A tiling space is naturally a dynamical system when equipped with the translation action (which is an $\mathbb{R}^2$ action as opposed to an action of $\mathbb{N}$ or $\mathbb{Z}$.)

Possible projects:

- Discuss an example of a tiling space (e.g., Penrose tilings, Robinson tilings, Pinwheel tilings, etc.). Discuss the topology on the tiling space. Discuss how properties of the $\mathbb{R}^2$-action relate to properties of the tilings in this particular case.

2.11. **Piecewise isometries.** Suppose we have a set X in the plane and two partitions of X into isometric pieces. Say $\{P_1, \ldots, P_n\}$ is one partition and $\{Q_1, \ldots, Q_n\}$ is another

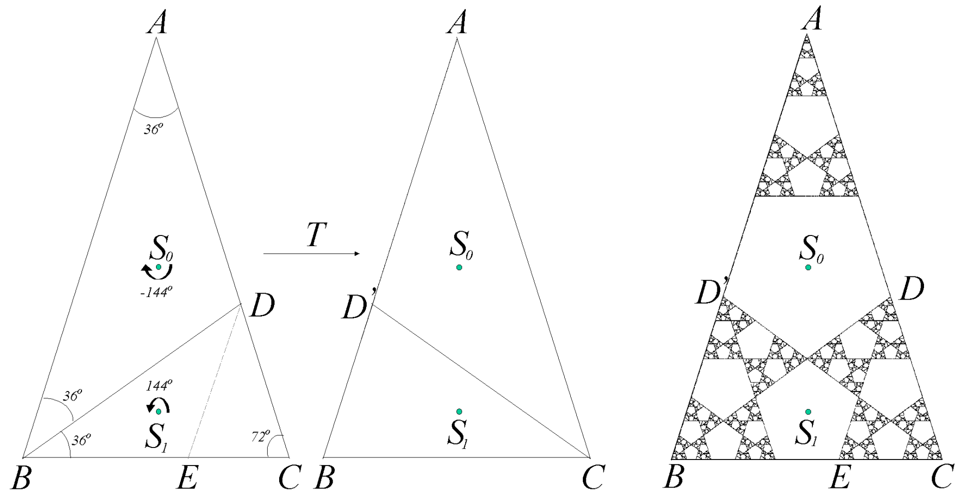

FIGURE 5. Left: An example of a piecewise isometry. Right: The limit set. This figure was taken from [Goe00].

and $\iota_i : P_i \to Q_i$ are isometries defined for $i \in \{1, \dots, n\}$. This information determines a dynamical system, $f : X \to X$, defined by $f(x) = \iota_i(x)$ when $x \in P_i$.

The orbit structure of such piecewise isometries can have some interesting fractal structure. See Figure 5.

Possible projects:

- Pick an example of a piecewise isometry. (For example, the map in Figure [**?**] is possible.) Describe how renormalization works and explains the self similar structure of the limit set. (Or explain other dynamical phenomena depending on the example.)

3. Additional references

[Sil16] Provides a brief elementary background on what ergodicity and equidistribution is. Available at `http://www.ams.org/publications/journals/notices/201601/rnoti-p26.pdf`

References

[BL06] Kurt Bryan and Tanya Leise, *The $25,000,000,000 eigenvector: The linear algebra behind Google*, SIAM Review **48** (2006), no. 3, 569–581.

[Goe00] Arek Goetz, *A self-similar example of a piecewise isometric attractor*, Dynamical systems (Luminy-Marseille, 1998) (2000), 248–258.

[Sil16] Cesar E Silva, *What is an ergodic transformation?*, Notices of the AMS **63** (2016), no. 1.