Dynamical Systems: Problem Set 1

This problem set concerns the quadratic maps of $\mathbb{R}$. A *quadratic map* is a map of the form

$$Q(x) = ax^2 + bx + c,$$

where $a, b, c \in \mathbb{R}$ are constants with $a \neq 0$. Note that $Q(x)$ is not invertible. The graph of $Q(x)$ is a parabola. The *critical point* of Q is the point x_c for which $Q'(x_c) = 0$.

You should be able to answer the following questions using a combinations of explicit calculations and cobweb plots for orbits.

1. The function Q is either concave up (the case $\lim_{x\to\pm\infty} Q(x) = +\infty$). Or concave down (the case $\lim_{x\to\pm\infty} Q(x) = -\infty$). Let $F(x) = -x$. Show that if Q is concave up, then the topologically conjugate quadratic map $F \circ Q \circ F^{-1}$ is concave down. (This says that to understand quadratic maps, it is enough to understand those that are concave down.)

2. Suppose that $R(x)$ is a concave down quadratic map with no fixed points. Describe the long term behavior of orbits: For any x, what is $\lim_{n\to+\infty} R^n(x)$?

3. Now suppose that Q is a concave up quadratic map with no fixed points. Explain how the two previous parts can be used to understand the long term behavior of orbits of Q without redoing your analysis.

4. Now consider the case when $Q(x)$ is concave down and has only one fixed point. Call the fixed point p. For any x, what is $\lim_{n\to+\infty} Q^n(x)$? (*Hint:* Let q be the point not equal to p so that $Q(q) = p$. How does the behavior of points in the interval $[p, q]$ differ from those outside?

5. The dynamics maps Q with two fixed points can be quite complex as we will see later in the course. We will content ourselves with putting Q into a standard form. A map of the form $L(x) = dx + e$ with $d \neq 0$ is called an *affine linear map*. Let $G(y) = L \circ Q \circ L^{-1}(y)$. Show:
   (a) The map G is quadratic.
   (b) If $y = L(x)$, then $G'(y) = Q'(x)$.
   (c) If $y = L(x)$, then x is a fixed point of Q if and only if y is a fixed point of G.

6. Suppose F is a quadratic map with two fixed points, with a critical point at $\frac{1}{2}$ (i.e., $F'(\frac{1}{2}) = 0$). Also suppose that zero is fixed and the other fixed point is closer to the critical point. Prove that $F'(0) > 1$. Set $\mu = F'(0) > 1$. Find an explicit formula for $F(x)$ in terms of x and μ.

7. Use the previous two parts to prove that any quadratic map $Q(x)$ with two fixed points is conjugate by an affine linear map to a map of the form F_μ for some $\mu > 1$.