Math A4500: Midterm 1 Study Guide

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Disclaimer. This test is just a recommendation of things to study. You may be asked about things that do not appear here. I recommend ensuring that you can do all the problems in the sections covered in the book.

Covered sections: We've covered sections A.1 and A.2 of the appendix, as well as sections I.1-I.5. We have also discussed topological conjugacy in class.
Definitions. You will be asked to define several terms on the test. The following is a list of terms which might appear.

class C^r, one-to one, onto, homeomorphism, C^r-diffeomorphism, limit point, closed set, open set, dense, forward orbit, backward orbit, orbit, periodic point, period, prime period (or least period), eventually periodic, forward and backward asymptotic, critical point, hyperbolic periodic point, multiplier, attracting periodic point, repelling periodic point, Cantor set, totally disconnected, perfect

You are expected to know the definitions given in the book. However, a better definition of dense is a subset of $U \subset S$ is *dense* in S if $S \subset \bar{U}$. Maps $f : X \to X$ and $g : Y \to Y$ are *topologically conjugate* if there is a homeomorphism $h : X \to Y$ so that $h \circ f(x) = g \circ h(x)$ for all $x \in X$.

Advanced Calculus. You should be able to use the state the basic theorems from calculus and be able to use them to prove dynamical results. The following are such theorems:

Intermediate value theorem, Mean value theorem

Theorems. Theorems and results from the book you should be able to prove.

- Proposition on page 390: If I is a closed and bounded interval, and $f : I \to I$ is continuous, then f has a fixed point.
- Proposition on page 390: If $f : I \to I$ is a continuous map of a closed and bounded interval I, and $|f'(x)| < 1$, then f has a unique fixed point and f contracts distances.
- Propositions on pages 33 and 35: Local behaviors near a hyperbolic fixed point.
- §5.1: Results about the Logistic map when $1 < \mu < 3$.
- Theorem on page 47: If $\mu > 2 + \sqrt{5}$ than Λ is a Cantor set.

Techniques. You should be able to use the various techniques we have developed for understanding particular dynamical systems acting on subsets of the real line. For instance, you should be able to:

- Find fixed and periodic points of maps of $\mathbb{R}$ and S^1.
- Understand the long term behaviors of points (as in the term *forward* and *backward asymptotic.*

- Prove that a subset of the real line is a Cantor set. This set could be constructed or dynamically determined.
- Use topological conjugacy to prove dynamical statements about a map. We discussed in class how f and g are topologically conjugate then they have similar dynamical properties.

Types of problems. There will be between four and six multi-part problems on the midterm. You'll have the full class period (100 minutes) to complete the midterm. Problems of the following forms are likely to appear:

- *Homework*: Problems similar to assigned homework problems.
- *Recall Something, then prove something.*: Problems that as you to recall a definition or a major result, and then use it in a proof.
- *Math comprehension:* The problem states a new definition, which may not have seen before, and asks you to use the definition in basic proofs.
- *Prove a result:* Problems which ask you to prove a result which is proved in the book. See the section titled "Theorems" above.

Practice problems. These are some problems I gave on midterms when I taught this class in the past. They do not cover all the topics mentioned so far.

1. Complete the following definitions:
   (a) Let $r \geq 0$ be an integer. A real valued function f defined on $X \subset \mathbb{R}$ is *of class* C^r if...
   (b) Let $X \subset \mathbb{R}$ and let $f : X \to X$ be a function. Also let $p \in X$ be a point of prime period n under f. Then, a point $x \in X$ is *forward asymptotic* to p if ...
2. Let $f : [a, b] \to [a, b]$ be a continuous function defined on a closed interval with $a < b$. Prove that f has a fixed point.
3. (a) Complete the following definition:
   Let $p \in \mathbb{R}$ be a point of prime period n under a real-valued function f. The point p is *hyperbolic* if ...
   (b) Let $f : [a, b] \to [a, b]$ be a C^1 function, and suppose that p is a hyperbolic fixed point with $|f'(p)| < 1$. Prove that there is an open interval U about p such that if $x \in U$, then
$$\lim_{n\to\infty} f^n(x) = p.$$

4. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a homeomorphism so that $f(x) > x$ for all $x \in \mathbb{R}$. Prove that $\lim_{n\to\infty} f^n(x) = +\infty$ for all $x \in \mathbb{R}$.

5. (a) Complete the following definition:
   Let $f : A \to A$ and $g : B \to B$ be two maps with $A, B \subset \mathbb{R}$. We say f and g are *topologically conjugate* if . . .

   (b) (8 points) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x + 1$ and define $h(x) = e^x$. Then h is a topological conjugacy from f to some map $g : Y \to Y$. Give a formula for the map g and find its domain Y.

6. (a) (10 points) Suppose I and J are intervals in $\mathbb{R}$ and $f : I \to I$ and $g : J \to J$ are continuous maps. Prove that if f and g are topologically conjugate via an orientation-preserving (increasing) homeomorphism $h : I \to J$, then if x_0 is a local minimum for f, then $h(x_0)$ is a local minimum for g.

   (b) (8 points) Use part (a) to prove that for any $c \neq 0$, the map $f(x) = x^2$ is not topologically conjugate to $g(x) = x^2 + c$ via an orientation-preserving homeomorphism $h : \mathbb{R} \to \mathbb{R}$.