THE LOGISTIC FAMILY: PARAMETERS LESS THAN ONE

W. PATRICK HOOPER

The Logistic family of maps is the collection of maps of the form

$$F_\mu : \mathbb{R} \to \mathbb{R}; \quad F_\mu(x) = \mu x(1-x).$$

The book does fairly careful study of these maps when the parameter μ is larger than one.

The case of $\mu = 1$ is special. Instead of having two fixed points, the map has a single fixed point at 0. (Because the other fixed point $p_\mu = \frac{\mu-1}{\mu}$ is also at 0. The graph is of F_1 is tangent to the diagonal at the origin. The dynamics are fairly simple, with orbits points in $[0, \frac{1}{2}]$ decreasing to zero. But it is of some interest, because 0 is not an attracting fixed point because points in $(-\infty, 0)$ still tend to $-\infty$.

Another even more special parameter appears when $\mu = 0$. So, of course $F_0(x) = 0$ for all x!

The other maps F_μ with $\mu < 1$ are actually "the same" as maps F_μ with $\mu > 1$. The way to make sense of how two maps have the same dynamics is via conjugacy. Recall that two maps $f, g : \mathbb{R} \to \mathbb{R}$ are *topologically conjugate* via a homeomorphism $\phi : \mathbb{R} \to \mathbb{R}$ if

$$\phi \circ f(x) = g \circ \phi(x) \quad \text{for all } x \in \mathbb{R}.$$

Here ϕ is called the *conjugacy* or *conjugating map*. We also briefly discussed stronger notions of conjugacy. We say f and g are C^k*-conjugate* if there is a a conjugacy as above with is C^k.

The following theorem gives an explicit C^∞-conjugacy between the maps of the form F_μ with $\mu < 1$ and the maps F_μ with $\mu > 1$.

Theorem 1. *Suppose $\mu < 1$ and $\mu \neq 0$. Then the affine linear map*

$$\phi(x) = \frac{1}{2} + \frac{\mu(x - \frac{1}{2})}{2-\mu}$$

conjugates the action of F_μ to the action of $F_{2-\mu}$. That is,

$$\phi \circ F_\mu(x) = F_{2-\mu} \circ \phi(x) \quad \textit{for all } x \in \mathbb{R}.$$

The theorem can be proved by a calculation, but we prefer to give a more conceptual proof. This argument also explains how you might find this conjugating map.

Having a conjugacy like this is equivalent to satisfying the identity

$$F_{2-\mu} = \phi \circ F_\mu \circ \phi^{-1}.$$

An affine linear map ϕ is simply a polynomial of degree one. Such maps have inverses which are also degree one polynomials. When you compose two polynomials, the degree of the composition is the product of the degrees. So, if ϕ is any affine-linear map, we know that $\phi \circ F_\mu \circ \phi^{-1}$ is a polynomial of degree two.

The Logistic family of maps can be defined as the collection of degree two polynomials with a fixed point at zero and with a critical point at $\frac{1}{2}$. When $\mu \neq 1$, the map F_μ has

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another fixed point at $p_\mu = \frac{\mu-1}{\mu}$. The composition $\phi \circ F_\mu \circ \phi^{-1}$ then has a fixed point at $\phi(p_\mu)$ because:

$$\phi \circ F_\mu \circ \phi^{-1}\big(\phi(p_\mu)\big) = \phi \circ F_\mu(p_\mu) = \phi(p_\mu).$$

Also, if ϕ is C^1, then $\phi(\frac{1}{2})$ must be a critical point for $\phi \circ F_\mu \circ \phi^{-1}$. This is because of the chain rule:

$$\frac{d}{dx}[\phi \circ F_\mu \circ \phi^{-1}(x)] = \phi'\big(F_\mu \circ \phi^{-1}(x)\big) \cdot F'_\mu\big(\phi^{-1}(x)\big) \cdot (\phi^{-1})'(x). \tag{1}$$

Note that when $x = \phi(\frac{1}{2})$, the middle term becomes $F'_\mu(\frac{1}{2}) = 0$. This proves that $\phi(\frac{1}{2})$ is a critical point for the $\phi \circ F_\mu \circ \phi^{-1}$.

From the above paragraph, if $\phi(x)$ is an affine linear map (of the form $\phi(x) = ax + b$) so that

$$\phi(p_\mu) = 0 \quad \text{and} \quad \phi(\tfrac{1}{2}) = \tfrac{1}{2},$$

then the conjugate map $\phi \circ F_\mu \circ \phi^{-1}$ is in the logistic family. There is only one affine linear map of this form, and it is the one given in the theorem.

Let $G(x) = \phi \circ F_\mu \circ \phi^{-1}(x)$. We have proved that $G(x)$ is in the logistic family, i.e. $G(x) = F_\nu$ for some $\nu \in \mathbb{R}$. We can recover ν by computing $G'(0)$. This is because $F'_\nu(0) = \nu$. Using our formula in equation 1, we have

$$G'(0) = \phi'\big(F_\mu \circ \phi^{-1}(0)\big) \cdot F'_\mu\big(\phi^{-1}(0)\big) \cdot (\phi^{-1})'(0).$$

Since ϕ is affine linear, the derivative of ϕ is constant, and is the multiplicative inverse of the derivative of ϕ^{-1}, which is also constant. Thus,

$$G'(0) = F'_\mu\big(\phi^{-1}(0)\big).$$

Since $\phi(p_\mu) = 0$, we know that $\phi^{-1}(0) = p_\mu$. Therefore, $G'(0) = F'_\mu(p_\mu)$. (This is also a consequence of the fact that a C^1 conjugacy sends periodic points to periodic points, and preserves the multiplier at those points.) We compute that $F'_\mu(x) = \mu(1 - 2x)$. Thus

$$G'(0) = F'_\mu(p_\mu) = \mu(1 - 2p_\mu) = \mu(1 - \frac{2(\mu-1)}{\mu}) = 2 - \mu.$$

This proves that $G = F_{2-\mu}$. (Note that this calculation doesn't work if $\mu = 0$ because we would be dividing by zero above.)

The City College of New York, New York, NY, USA 10031
Email address: whooper@ccny.cuny.edu