THE LOGISTIC FAMILY WITH $2 < \mu < 3$

W. PATRICK HOOPER

Let $2 < \mu < 3$ and $2 < \nu < 3$. Consider the two Logistic maps:

$$F(x) = \mu x(1-x) \quad \text{and} \quad G(x) = \nu x(1-x).$$

The purpose of this note is to prove:

Theorem 1. *The maps F and G are topologically conjugate on $[0,1]$. That is, there is a homeomorphism $h : [0,1] \to [0,1]$ so that*

$$h \circ F(x) = G \circ h(x) \quad \textit{for all } x \in [0,1]. \tag{1}$$

It is actually true that F and G are topologically conjugate on all of $\mathbb{R}$, but we will not bother to prove this as the dynamics outside of $[0,1]$ are less interesting. (F and G are expanding on $(-\infty, 0]$ with a fixed point at zero.)

Observe that the following two restrictions of F are homeomorphisms onto their image:

$$F_L = F|_{(-\infty,\frac{1}{2}]} : (-\infty, \tfrac{1}{2}] \to (-\infty, \frac{\mu}{4}] \quad \text{and} \quad F_R = F|_{[\frac{1}{2},+\infty)} : [\tfrac{1}{2}, +\infty) \to (-\infty, \frac{\mu}{4}].$$

The map F_L is increasing while F_R is decreasing. We will make use of F_L^{-1} and F_R^{-1}. We similarly define G_L^{-1} and G_R^{-1}.

First we note some conditions that must be satisfied by h.

Proposition 2. *If h satisfies (1), then:*

(1) $h(0) = h(0)$ and $h(p_\mu) = p_\nu$ where $p_\mu = \frac{\mu-1}{\mu}$ and $p_\nu = \frac{\nu-1}{\nu}$.
(2) h is monotone increasing.
(3) $h(\frac{\mu}{4}) = \frac{\nu}{4}$.
(4) $h(\frac{1}{2}) = \frac{1}{2}$.
(5) For $x \leq \frac{\mu}{4}$ we have $h \circ F_R^{-1}(x) = G_R^{-1} \circ h(x)$ and $h \circ F_L^{-1}(x) = G_L^{-1} \circ h(x)$.

Proof. To see (1) recall that topological conjugacies map attracting fixed points to attracting fixed points and repelling fixed points to repelling fixed points. The two fixed points of F are 0 and p_μ. The point 0 is repelling since $F'(0) = \mu > 2$ while p_μ is attracting since $-1 < F'(p_\mu) = 2 - \mu < 0$. Similarly 0 is an repelling fixed point for G while p_ν is an attracting. So we must have $h(0) = 0$ and $h(p_\mu) = p_\nu$.

To see (2) recall that homeomorphisms of $\mathbb{R}$ are either monotone increasing or monotone decreasing. Since $0 < p_\mu$ and $h(0) = 0 < p_\nu = h(p_\mu)$, the map h must be monotone increasing.

To see (3) observe that the range of F is $(-\infty, \frac{\mu}{4}]$ and the range of G is $(-\infty, \frac{\mu}{4}]$. From the conjugacy equation, we must have $h \circ F(\mathbb{R}) = G \circ h(\mathbb{R})$. Since h is a homeomorphism of $\mathbb{R}$, $h(\mathbb{R}) = \mathbb{R}$. Thus $h \circ F(\mathbb{R}) = G(\mathbb{R})$. Then since h is monotone increasing, $h\big(\max F(\mathbb{R})\big) = \max G(\mathbb{R})$. But $\max F(\mathbb{R}) = \frac{\mu}{4}$ and $\max G(\mathbb{R}) = \frac{\nu}{4}$.

Date: March 3, 2017.

To see (4) observe that $y = \frac{1}{2}$ is the unique point satisfying $G(y) = \frac{\nu}{4}$. Observe that

$$G \circ h(\frac{1}{2}) = h \circ F(\frac{1}{2}) = h(\frac{\mu}{4}) = \frac{\nu}{4}.$$

Thus $G \circ h(\frac{1}{2}) = \frac{\nu}{4}$. From the uniqueness property of $y = \frac{1}{2}$ we have $h(\frac{1}{2}) = \frac{1}{2}$.

We will prove that $x \leq \frac{\mu}{4}$ implies $h \circ F_R^{-1}(x) = G_R^{-1} \circ h(x)$. The other statement is similar. Fix x. Let $x' = F_R^{-1}(x)$. Then $x' \geq \frac{1}{2}$ and $F(x') = x$. Set $y = h(x)$ and $y' = h(x')$. From the conjugacy equation we have

$$y = h(x) = h \circ F(x') = G \circ h(x') = G(y').$$

Then since h is monotone increasing and $h(\frac{1}{2}) = \frac{1}{2}$ we know $y' \geq \frac{1}{2}$. Since $y' \geq \frac{1}{2}$ and $y = G(y')$ we have $y' = G_R^{-1}(y)$. Putting it all together we have

$$h \circ F_R^{-1}(x) = h(x') = y' = G_R^{-1}(y) = G_R^{-1} \circ h(x).$$

□

It is not too hard to convince yourself that the interval $I = [\frac{1}{2}, F_R^{-1}(\frac{1}{2})]$ is F-invariant, i.e., $F(I) \subset I$. This can be done by direct computation. We will give a different proof. Actually F is contracting on I and we will use this to show that I is F-invariant. This contracting behavior is also the key which allows us to produce our topological conjugacy. This contracting behavior is proved in the Lemma below, and we establish F-invariance as a corollary. We also define $J = [\frac{1}{2}, G_R^{-1}(\frac{1}{2})]$ which we will show is G-invariant. It has similar properties.

Lemma 3. *We have* $0 \leq (F^2)'(x) < 1$ *on* I *and* $(G^2)'(x) < 1$ *on* J.

Proof. It is enough to prove this for F with $2 < \mu < 3$. From the chain rule

$$(F^2)'(x) = F'(x)F'\big(F(x)\big).$$

Note in particular that this implies that $(F^2)'$ is zero on the endpoints of I. We will find the maximum of $(F^2)'$ on I, and now we know this maximum does not occur at the endpoints. By computation we see

$$(F^2)'(x) = \mu(1 - 2x) \cdot \mu\big(1 - 2\mu x(1 - x)\big) = \mu^2(-4\mu x^3 + 6\mu x^2 - 2\mu x - 2x + 1).$$

Local minima and maxima occur at critical points of $(F^2)'$ so we compute

$$(F^2)''(x) = \mu^2(-12\mu x^2 + 12\mu x - 2\mu - 2).$$

Using the quadratic formula we see that the critical points are

$$c_\pm = \frac{1}{2} \pm \frac{\sqrt{3\mu^2 - 6\mu}}{6\mu}.$$

Since $(F^2)'$ has equal values at the endpoints of I, there must be a critical point in I. This must be c_+ since $c_- < \frac{1}{2}$. We now evaluate

$$(F^2)'(c_+) = \frac{\mu(\mu - 2)\sqrt{3\mu(\mu - 2)}}{9}.$$

We observe that $2 < \mu < 3$, $0 < \mu - 2 < 1$, and $0 < \sqrt{3\mu(\mu - 2)} < 3$ so that $0 < (F^2)'(c_+) < 1$. Thus c_+ must be the maximum of $(F^2)'$ on this interval and the endpoints realize a minimum of zero so:

$$0 \leq (F^2)'(x) \leq (F^2)'(c_+) < 1.$$

□

Corollary 4. *The interval I is F-invariant and the interval J is G-invariant.*

Proof. It suffices to prove the statement for F for an arbitrary μ with $2 < \mu < 3$. Since the left endpoint of I is $\frac{1}{2}$, we know that F is monotone decreasing on I (i.e., F is order reversing). Then

$$F(I) = F\big([\tfrac{1}{2}, F_R^{-1}(\tfrac{1}{2})]\big) = [F \circ F_R^{-1}(\tfrac{1}{2}), F(\tfrac{1}{2})] = [\tfrac{1}{2}, F(\tfrac{1}{2})].$$

So, it suffices to prove that $F(\frac{1}{2}) \leq F_R^{-1}(\frac{1}{2})$. Suppose to the contrary that $F(\frac{1}{2}) > F_R^{-1}(\frac{1}{2})$. Then since F is monotone decreasing on $[\frac{1}{2}, \infty)$ we know that

$$F^2(\tfrac{1}{2}) < F \circ F_R^{-1}(\tfrac{1}{2}) = \tfrac{1}{2}.$$

This describes F^2 applied to the left endpoint, and for the right we have:

$$F^2\big(F_R^{-1}(\tfrac{1}{2})\big) = F(\tfrac{1}{2}) > F_R^{-1}(\tfrac{1}{2})$$

by hypothesis. Now we apply the Mean Value Theorem to the interval I. We see there is a point $x \in I$ so that

$$(F^2)'(x) = \frac{F^2\big(F_R^{-1}(\frac{1}{2})\big) - F^2(\frac{1}{2})}{F_R^{-1}(\frac{1}{2}) - \frac{1}{2}} > \frac{F_R^{-1}(\frac{1}{2}) - \frac{1}{2}}{F_R^{-1}(\frac{1}{2}) - \frac{1}{2}} = 1.$$

But this contradicts the lemma above. □

The above shows that $F|_I$ and $G|_J$ are contractions (which follows from the Mean Value theorem). Recall that contractions leaving invariant a compact set has a fixed point in that set, so we have a soft argument that $p_\mu \in I$ and $p_\nu \in J$. Then points move toward p_μ and p_ν:

Corollary 5.

- *If $x \in I$ and $x < p_\mu$ then $x < F^2(x) < p_\mu$. If $y \in J$ and $y < p_\nu$ then $y < G^2(y) < p_\nu$.*
- *If $x \in I$ and $p_\mu < x$ then $p_\mu < F^2(x) < x$. If $y \in J$ and $p_\nu < y$ then $y < G^2(y) < p_\nu$.*
- *If $x \in I$ then $\lim_{n\to+\infty} F^n(x) = p_\mu$. If $y \in J$ then $\lim_{n\to+\infty} G^n(x) = p_\nu$.*

Discussion of proof. The first two statements follow from the Mean Value Theorem. The last statement is similar to results discussed in class and follows from the first two statements. Alternately, we can see from the Mean Value Theorem and the fact that continuous functions on compact sets attain there maximum (applied to F' on I) that F^2 contracts distances to p_μ by a multiplicative factor of some $\lambda < 1$. □

As a special case of the above we see that $F(\frac{1}{2}) < F_R^{-1}(\frac{1}{2})$, since $F(\frac{1}{2}) = F^2 \circ F_R^{-1}(\frac{1}{2})$. Consider the interval $I_0 = [F(\frac{1}{2}), F_R^{-1}(\frac{1}{2})]$. Observe that it follows from the above that the forward orbits of the intervals have disjoint interiors and

$$\bigcup_{n \geq 0} F^n(I_0) = I \smallsetminus \{p_\mu\}.$$

The endpoints of these image intervals are the adjacent points other than p_μ in the inequalities:

$$\tfrac{1}{2} < F^2(\tfrac{1}{2}) < F^4(\tfrac{1}{2}) < \ldots < p_\mu < \ldots < F^3(\tfrac{1}{2}) < F(\tfrac{1}{2}) < F_R^{-1}(\tfrac{1}{2}).$$

Define $J_0 = [G(\frac{1}{2}), G_R^{-1}(\frac{1}{2})]$. Similar statements hold.

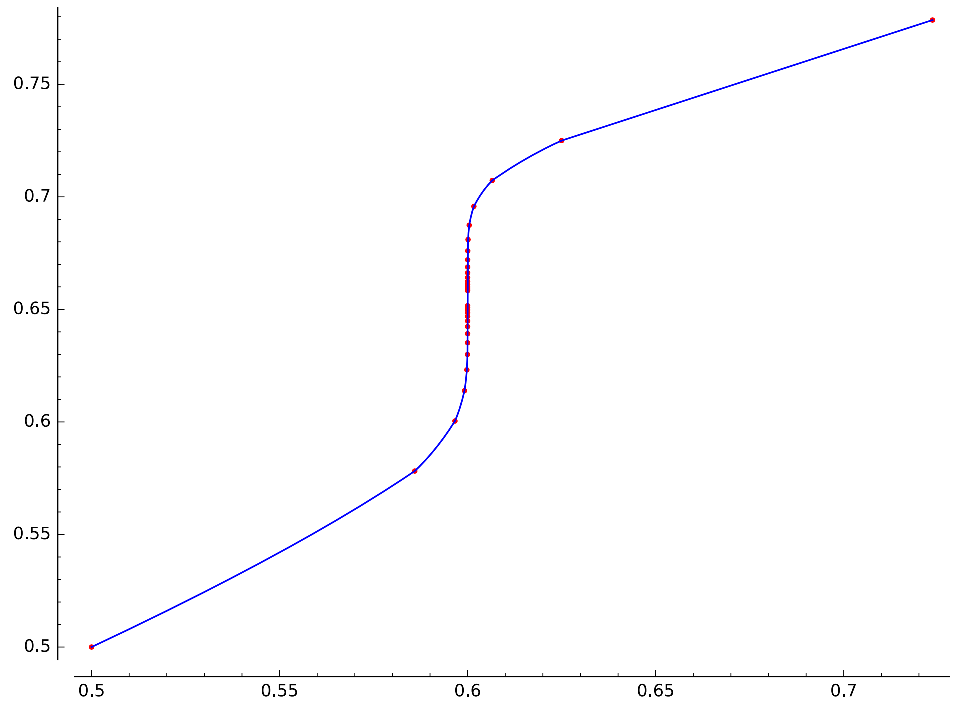

FIGURE 1. The graph of h_I when $\mu = 2.5$ and $\nu = 2.9$ and h_0 is taken as in (2). The marked points pairs (x, y) where $x = F^n \circ F_R^{-1}(\frac{1}{2})$ and $y = G^n \circ G_R^{-1}(\frac{1}{2})$ for some $n \geq 0$. Do you know why h_I can not possibly be differentiable at p_μ?

Now we will begin defining our topological conjugacy. Begin with a monotone increasing homeomorphism $h_0 : I_0 \to J_0$. For example, a natural map like that is the affine (linear) map:

$$h_0(x) = G(\tfrac{1}{2}) + \frac{G_R^{-1}(\frac{1}{2}) - G(\frac{1}{2})}{F_R^{-1}(\frac{1}{2}) - F(\frac{1}{2})}\big(x - F(\tfrac{1}{2})\big). \tag{2}$$

We will extend h_0 to a homomorphism $h_I : I \to J$ so that $h_I \circ F(x) = G \circ h_I(x)$ for $x \in I$. Observe that if $x \in I$ then either $x = p_\mu$ or there is an $n \geq 0$ so that $F_R^{-n}(x) \in I_0$. Using this we define:

$$h_I(x) = \begin{cases} p_\nu & \text{if } x = p_\mu \\ G_R^n \circ h_0 \circ F_R^{-n}(x) & \text{if } F_R^{-n}(x) \in I_0. \end{cases}$$

An example graph of h_I is shown in Figure 1. We have:

Proposition 6. *The map $h_I : I \to J$ is well defined and is a topological conjugacy from $F|_I$ to $G|_J$.*

Sketch of proof. It is not difficult to observe that h_I is strictly increasing on intervals of the form $F^n(I_0)$, and that h_I sends the endpoints of I to the endpoints of J (respecting their order). The endpoints of the intervals $F^n(I_0)$ coincide with the orbit $F^n \circ F_R^{-1}(\frac{1}{2})$ and this can be used to show that h_I is well defined and h_I is continuous at these endpoints. The last statement of Corollary 5 implies that h_I is continuous at p_μ as well. Thus h_I is strictly increasing and is therefore a homeomorphism.

We must check the topological conjugacy equation, i.e., that $h_I \circ F(x) = G \circ h_I(x)$ for $x \in I$. Based on our definition that $h_I(p_\mu) = p_\nu$, this is clearly true for the fixed point p_μ. Now suppose $x \neq p_\mu$. As mentioned above, there is an n so that $F_R^{-n}(x) \in I_0$. Then we defined $h_I(x) = G_R^n \circ h_0 \circ F_R^{-n}(x)$. To evaluate the equation, we also need to know $h_I \circ F(x)$. Observe that because $F_R^{-n}(x) \in I_0$, we have $F_R^{-n-1}\big(F(x)\big) \in I_0$ so our definition gives

$$h_I \circ F(x) = G_R^{n+1} \circ h_0 \circ F_R^{-n-1}\big(F(x)\big) = G_R^{n+1} \circ h_0 \circ F_R^{-n}(x) = G\big(h_I(x)\big),$$

completing the check. □

Now consider the interval $I_1 = [0, F_R^{-1}(\frac{1}{2})]$ and $J_1 = [0, G_R^{-1}(\frac{1}{2})]$. As $\mu > 2$, we have:

$$F(x) > x \quad \text{when } 0 < x \leq \tfrac{1}{2}.$$

It then follows that if $0 < x < \frac{1}{2}$ there is a smallest $n > 0$ so that $F^n(x) \in I$. We define

$$h_1 : I_1 \to J_1; \quad h_1(x) = \begin{cases} 0 & \text{if x=0,} \\ h_I(x) & \text{if } x \in I, \\ G_L^{-n} \circ h_I \circ F^n(x) & \text{if } 0 < x < \frac{1}{2}. \end{cases}$$

An example graph is given in Figure 2.

Proposition 7. *The map $h_1 : I_1 \to J_1$ is well defined and is a topological conjugacy from $F|_{I_1}$ to $G|_{J_1}$.*

Sketch of proof. Recall that the maximum of F is contained in the interval I. It follows that $F_L^{-1}(I) = \big[F_L^{-1}(\frac{1}{2}), \frac{1}{2}\big]$. Then $F_L^{-n}(I) = \big[F_L^{-n}(\frac{1}{2}), F_L^{-n+1}(\frac{1}{2})\big]$ for $n \geq 1$. By construction the maps are continuous on the interiors of these intervals. The points $F_L^{-n}(\frac{1}{2})$ are both the left endpoint of $F_L^{-n}(I)$ and the right endpoint of $F_L^{-n-1}(I)$. To see continuity at $x_0 = F_L^{-n}(\frac{1}{2})$ observe that

$$\lim_{x \to x_0^+} h_I(x_n) = G_L^{-n} \circ h_I \circ F^n(x_0) = G_L^{-n} \circ h_I(\tfrac{1}{2}) \quad \text{and}$$

$$\lim_{x \to x_0^-} h_I(x_n) = G_L^{-n-1} \circ h_I \circ F^{n+1}(x_0) = G_L^{-n-1} \circ h_I \circ F(\tfrac{1}{2}) = G_L^{-n-1} \circ G \circ h_I(\tfrac{1}{2}) = G_L^{-n}(\tfrac{1}{2}),$$

where in the second line we used that h_I is a conjugacy from $F|_I$ to $G|_J$. Finally, since $0 < x \leq \frac{1}{2}$ implies $0 < F(x) < x$, we know that $\lim_{n\to\infty} F_L^{-n}(x) = 0$ whenever $0 < x \leq \frac{1}{2}$. This observation for F and G tells us that $\lim_{x\to 0^+} h_1(x) = 0$ so that h_1 is continuous at 0.

The conjugacy equation can be checked and follows the argument given in the proof of Proposition 6. □

It remains to extend h_1 to $h : [0,1] \to [0,1]$. Recall that an *F-invariant set* is a set S so that $F(S) \subset S$. How we should extend is determined by the following observation:

Proposition 8. *If $h : [0,1] \to [0,1]$ is an orientation preserving (monotone increasing) topological conjugacy from F to G then $h(1-x) = 1 - h(x)$ for any $x \in [0,1]$.*

Proof. First suppose $x = \frac{1}{2}$. In this case the equation $h(1-x) = 1 - h(x)$ is equivalent to $h(\frac{1}{2}) = \frac{1}{2}$ which we already know since $\frac{1}{2}$ is the only critical point. The key observation in the remaining cases is that

$$F(x) = F(1-x) \quad \text{and} \quad G(x) = G(1-x).$$

Suppose $x < \frac{1}{2}$. Then

$$1 - x = F_R^{-1} \circ F(x).$$

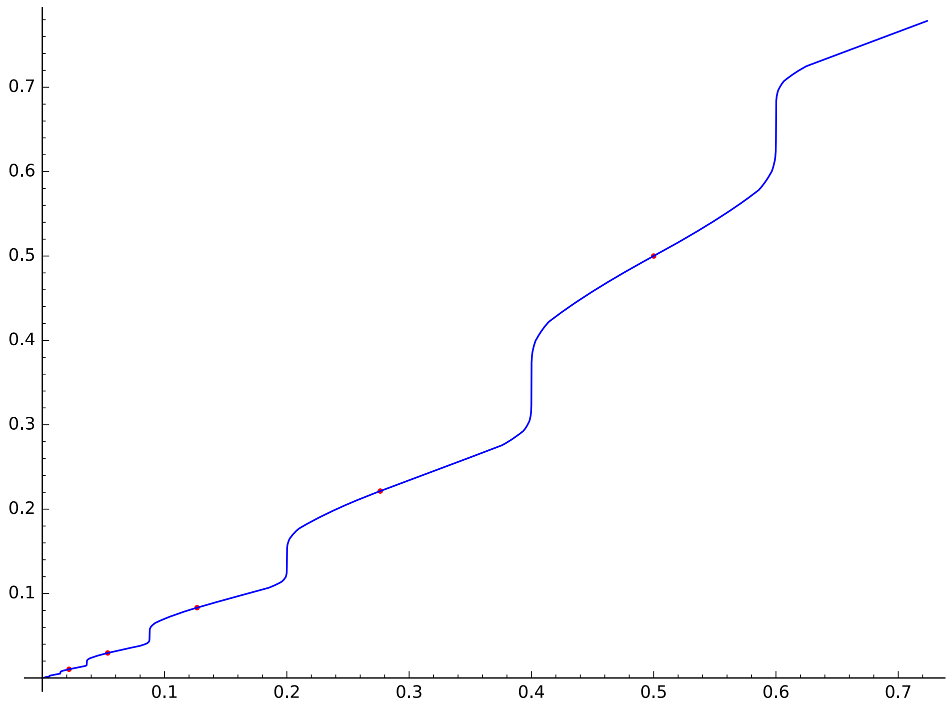

FIGURE 2. The graph of h_1 when $\mu = 2.5$ and $\nu = 2.9$ and h_0 is taken as in (2). Points $F_L^{-n}(\frac{1}{2})$ with $0 \leq n \leq 4$ are marked.

From Proposition 2 we know that $h(x) < \frac{1}{2}$ as well. Then we also have $1 - h(x) = G_R^{-1} \circ G(x)$. Now recall the conjugacy equation and that we must have $h \circ F_R^{-1} = G_R^{-1} \circ h$ from Proposition 2. We have

$$h(1-x) = h \circ F_R^{-1} \circ F(x) = G_R^{-1} \circ h \circ F(x) = G_R^{-1} \circ G \circ h(x) = 1 - h(x).$$

The case of $x > \frac{1}{2}$ is similar. □

We note in particular that this is already true our map h_1 in the following sense:

Proposition 9. *If both x and $1-x$ lie in I_1 then $h_1(1-x) = 1 - h_1(x)$.*

Proof. Since $h(\frac{1}{2}) = \frac{1}{2}$, this is obviously true for $x = \frac{1}{2}$.

The conclusion is symmetric in swapping x with $1-x$, so we may assume without loss of that $x > \frac{1}{2}$. Then $x = F_R^{-1} \circ F(x)$ and $1 - x = F_L^{-1} \circ F(x)$. Then $1 - x \notin I$ but $F(1-x) = F(x) \in I$. So by definition of h_1 we have

$$h_1(1-x) = G_L^{-1} \circ h_I \circ F(x).$$

Then because h_I was a topological conjugacy,

$$h_1(1-x) = G_L^{-1} \circ G \circ h_I(x).$$

Then because $h_I(x) \in J$ and $x > \frac{1}{2}$ we have $h_I(x) > \frac{1}{2}$. This means that

$$h_1(1-x) = 1 - h_I(x) = 1 - h_1(x)$$

as desired. □

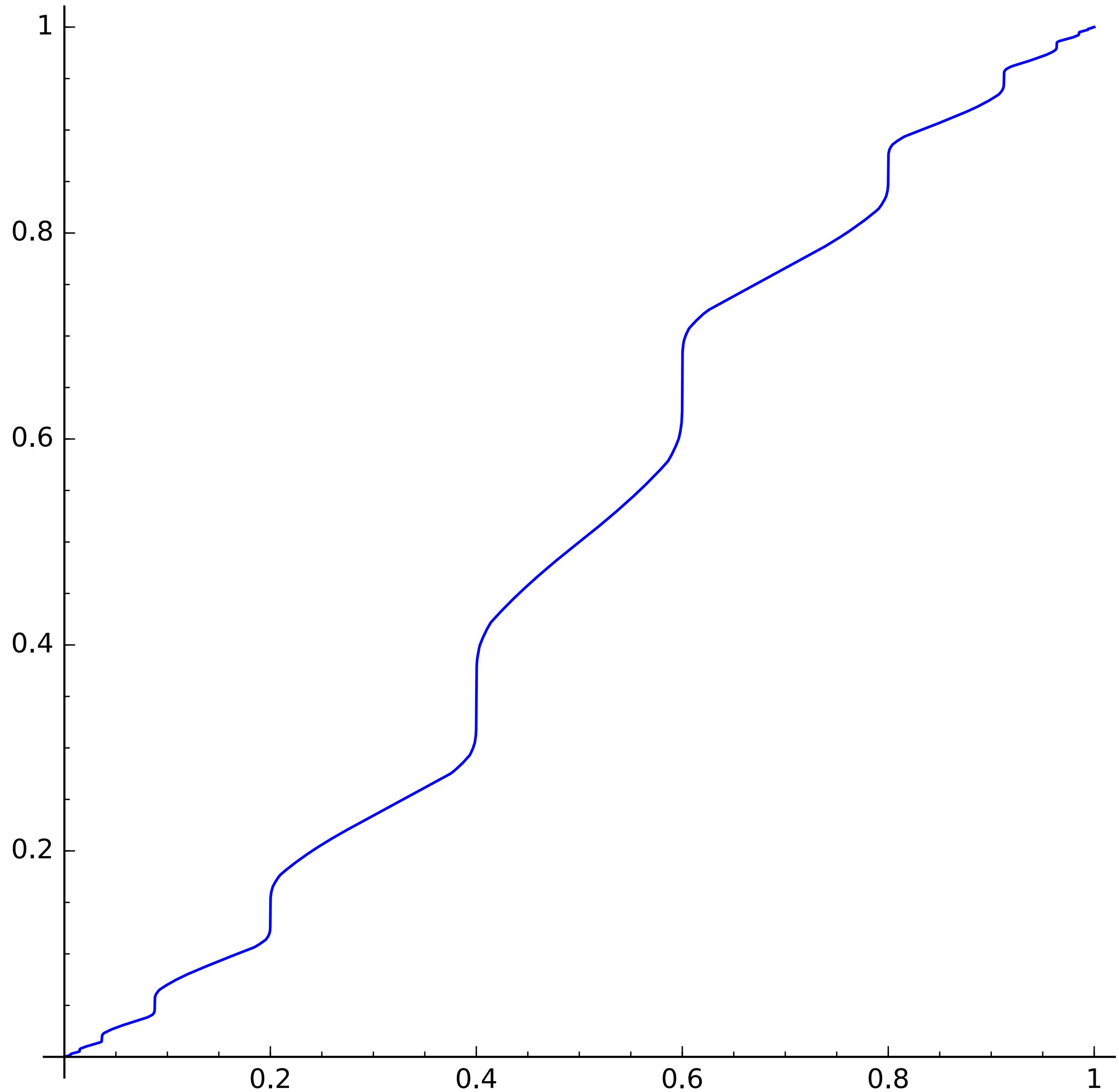

FIGURE 3. The graph of h when $\mu = 2.5$ and $\nu = 2.9$ and h_0 is taken as in (2).

Proposition 8 tells us how we must extend h_1 to $h : [0,1] \to [0,1]$. Namely, we define

$$h : [0,1] \to [0,1]; \quad h(x) = \begin{cases} h_1(x) & \text{if } x \in I_1, \\ 1 - h_1(1-x) & \text{if } x \notin I_1. \end{cases}$$

The second line must be how we extend since when $x \notin I_1$ we have $1 - x \in I_1$ and so

$$h(x) = h\big(1 - (1-x)\big) = 1 - h(1-x)$$

using Proposition 8. An example graph of h is shown in Figure 3

To complete the proof of the Theorem we need to show:

Proposition 10. *The map h is a topological conjugacy from F to G on* $[0,1]$.

Proof. First we need to see that h is a homeomorphism. We already know it is strictly increasing on I_1. Also our definition on $[0,1] \setminus I_1$ can be observed to be strictly increasing. Because of our definition of h in pieces, there is a possible discontinuity at the right endpoint of I_1 which we need to rule out. This is ruled out by Proposition 9 since this proposition tells us that the maps $x \mapsto h_1(x)$ and $x \mapsto 1 - h_1(1-x)$ must agree on $I = [\frac{1}{2}, F_L^{-1}(\frac{1}{2})]$. So, h is continuous.

It remains to check the conjugacy equation. Fix an $x \in [0,1]$. We need to show that $h \circ F(x) = G \circ h(x)$. If $x \in I_1$, then this already follows from Proposition 7. So we may assume that $x > F_R^{-1}(\frac{1}{2}) > \frac{1}{2}$. In this case $h(x) = 1 - h_1(1-x)$, so

$$G \circ h(x) = G\big(1 - h_1(1-x)\big) = G \circ h_1(1-x) = h_1 \circ F(1-x),$$

since h_1 was a conjugacy on I_1 by Proposition 7. Then $F(1-x) = F(x)$ so

$$G \circ h(x) = h_1 \circ F(1-x) = h_1 \circ F(x).$$

But we know $h = h_1$ on the domain of h_1 so

$$G \circ h(x) = h \circ F(x)$$

completing the proof. □

The City College of New York, New York, NY, USA 10031
E-mail address: whooper@ccny.cuny.edu