CANTOR SETS IN THE LOGISTIC FAMILY

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We consider the logistic map $F_\mu(x) = \mu x(1-x)$ when $\mu > 4$. Define

$$\Lambda = \{x \in [0,1] \ : \ F_\mu^n(x) \in [0,1] \text{ for all } n \geq 0\}.$$

We will prove the following theorem:

Theorem 1. *If $\mu > 2 + \sqrt{5}$, then Λ is a Cantor set.*

The goal of this note is to give a more explicit proof of this result than what is offered in Devaney's book.

We will now give an equivalent description of Λ. Define the set

$$A_0 = \{x \in [0,1] \ : \ F_\mu(x) \notin [0,1]\}.$$

Then A_0 is an open interval:

$$A_0 = \left(\frac{1}{2} - \frac{\sqrt{\mu-4}}{2\sqrt{\mu}}, \frac{1}{2} + \frac{\sqrt{\mu-4}}{2\sqrt{\mu}}\right).$$

We can then inductively define A_n for all integers $n \geq 0$ inductively according to the rule that $A_{n+1} = F_\mu^{-1}(A_n)$. (Here we are using the *preimage*: $F_\mu^{-1}(A_n) = \{x : \ F_\mu(x) \in A_n\}$.) Then we have

$$\Lambda = [0,1] \setminus \bigcup_{n=0}^{\infty} A_n.$$

The reason for taking $\mu > 2 + \sqrt{5}$ is given by the following result:

Proposition 2. *If $\mu > 2 + \sqrt{5}$, then $|F_\mu'(x)| > 1$ for all $x \in [0,1] \setminus A_0$.*

Proof. From the above, we have

$$[0,1] \setminus A_0 = I_0 \cup I_1 \quad \text{where} \quad I_0 = \left[0, \frac{1}{2} - \frac{\sqrt{\mu-4}}{2\sqrt{\mu}}\right] \quad \text{and} \quad I_1 = \left[\frac{1}{2} + \frac{\sqrt{\mu-4}}{2\sqrt{\mu}}, 1\right].$$

Observe that the derivative $F_\mu'(x) = \mu(1-2x)$ is monotone decreasing and positive on $[0, \frac{1}{2})$. So, the smallest value F_μ' takes on I_0 is at the right endpoint. We compute

$$F_\mu'\left(\frac{1}{2} - \frac{\sqrt{\mu-4}}{2\sqrt{\mu}}\right) = \mu(\sqrt{\mu-4} \cdot \sqrt{\mu}) = \sqrt{\mu^2 - 4\mu}.$$

Observe that the function $\mu \mapsto \mu^2 - 4\mu$ is increasing on the interval $[2, +\infty)$. So because $\mu > 2 + \sqrt{5}$, we have

$$\sqrt{\mu^2 - 4\mu} > \sqrt{(2+\sqrt{5})^2 - 4(2+\sqrt{5})} = 1.$$

We've shown that $|F_\mu'| > 1$ on I_0. But observe that I_1 is the image of I_0 under the map $x \mapsto 1-x$, and because $F_\mu'(1-x) = -F_\mu'(x)$, we see that $F_\mu'| > 1$ on $I_0 \cup I_1$. □

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The relevance of the derivative is given by the following, which follows from the fundamental theorem of calculus.

Proposition 3. *If f is a real-valued function which is differentiable and monotone on an interval $J \subset \mathbb{R}$, then*

$$\textit{length } f(J) = \int_J |f'(x)| \, dx.$$

These two propositions are enough to prove our Theorem.

Proof of Theorem 1. Suppose that $\mu > 2 + \sqrt{5}$. We will prove that Λ is closed, totally disconnected, and perfect.

1. We will prove that Λ is closed. Observe that A_0 is open. Recall that $A_{n+1} = F_\mu^{-1}(A_n)$. Since F_μ is continuous, $F_\mu^{-1}(U)$ is open for every open set $U \subset \mathbb{R}$. Therefore, we can see by induction that A_n is open for all n. An arbitrary union of open sets is open, so $\bigcup_{n=0}^{\infty} A_n$ is open. Then $\Lambda = [0,1] \smallsetminus \bigcup_{n=0}^{\infty} A_n$ must be closed.

2. We will prove that Λ is totally disconnected, that is Λ does not contain any (non-degenerate) intervals. By proposition 2, there is a $\lambda > 1$ so that $|F_\mu'(x)| \geq \lambda$ for all $x \in [0,1] \smallsetminus A_0$. (This uses the fact that a continuous function on a closed and bounded subset of $\mathbb{R}$ attains its maximum. We can take λ to be the maximal value of $|F_\mu'(x)|$ over $x \in [0,1] \smallsetminus A_0$.)

Suppose Λ does contain an interval J_0 of positive length. Then, inductively define J_n for all $n \geq 0$ according to the rule that $J_{n+1} = F_\mu(J_n)$. This makes $J_n = F_\mu^n(J_0)$ for all n. Since $J_0 \subset \Lambda$, we know that $J_n \subset \Lambda$ for all n. Then, it must be that each J_n lies in one of the two closed intervals making up $[0,1] \smallsetminus A_0$. Since F_μ is differentiable and monotone on each of these intervals, we can use Proposition 3 to see that

$$\text{length } J_{n+1} = \int_{J_n} |F_\mu'(x)| \, dx \geq \int_{J_n} \lambda \, dx = \lambda \cdot \text{length } J_n.$$

It follows that the length of J_n grows exponentially. But this is impossible since $J_n \subset [0,1]$ for all n.

3. We will prove that Λ is perfect. It is enough to prove that for all $x_0 \in \Lambda$ and all $\epsilon > 0$ there is a $y \in \Lambda$ so that $x_0 \neq y$ and $|x_0 - y| < \epsilon$.

The set $[0,1] \smallsetminus A_0$ consists of two closed intervals,

$$I_0 = \left[0, \frac{1}{2} - \frac{\sqrt{\mu - 4}}{2\sqrt{\mu}}\right] \quad \text{and} \quad I_1 = \left[\frac{1}{2} + \frac{\sqrt{\mu - 4}}{2\sqrt{\mu}}, 1\right].$$

The restrictions of F_μ to I_0 and I_1,

$$F_\mu|_{I_0} : I_0 \to [0,1] \quad \text{and} \quad F_\mu|_{I_1} : I_1 \to [0,1],$$

are homeomorphisms, so they have inverse maps

$$h_0 : [0,1] \to I_0 \quad \text{and} \quad h_1 : [0,1] \to I_1, \quad \text{respectively.}$$

Observe that if $x \in \Lambda$ then $h_0(x) \in \Lambda$ and $h_1(x) \in \Lambda$. This is because

$$x = F_\mu\big(h_0(x)\big) = F_\mu\big(h_1(x)\big).$$

So, the orbits of $h_0(x)$ and $h_1(x)$ follow the orbit of x.

By Proposition 3, we know that the restrictions $F_\mu|_{I_0}$ and $F_\mu|_{I_1}$ expand distance by a factor of $\lambda > 1$. So, h_0 and h_1 each contract distances by a factor of λ. That is, if $a, b \in [0,1]$, then

$$|h_0(a) - h_0(b)| \leq \frac{1}{\lambda}|a-b| \quad \text{and} \quad |h_1(a) - h_1(b)| \leq \frac{1}{\lambda}|a-b|.$$

Choose any $x_0 \in \Lambda$. Then the orbit $\{x_n = F_\mu^n(x_0) \ : \ n \geq 0\}$ is contained in Λ for all $n \geq 0$. Since $\Lambda \subset I_0 \cup I_1$, there is a sequence $\langle s_n \in \{0,1\} \ : \ n \geq 0\rangle$ so that

$$F_\mu^n(x) \in I_{s_n} \quad \text{for all } n \geq 0.$$

In particular, notice that since $F_\mu(x_n) = x_{n+1}$, we know that $x_n = h_{s_n}(x_{n+1})$, since $x_n \in I_{s_n}$ and the image of x_n under F_μ is x_{n+1}. It then follows that

$$x_0 = h_{s_0} \circ \ldots \circ h_{s_{n-2}} \circ h_{s_{n-1}}(x_n).$$

Now pick any $\epsilon > 0$. Then there is an n so that $\frac{1}{\lambda^n} < \epsilon$. Now choose $y' = 0$ or $y' = 1$, so that $y' \neq x_n$. Note that both points lie in Λ, since they map to zero which is fixed by F_μ. Define

$$y = h_{s_0} \circ \ldots \circ h_{s_{n-2}} \circ h_{s_{n-1}}(y').$$

Since each map is a homeomorphism, we know that $x_0 \neq y$. Also these maps contract distances by $\frac{1}{\lambda}$. So

$$|x_0 - y| \leq \frac{1}{\lambda^n}|x_n - y'| \leq \frac{1}{\lambda^n} < \epsilon.$$

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