Math 32404: Advanced Calculus II: Practice for Midterm 1

Topics Covered:

- Metric Spaces and Euclidean Spaces (Ross §13, Folland §1.1-1.2)
- Limits, Continuity, Sequences, and Completeness (Ross §13 and §21, Folland §1.3-1.5)
- Compactness (Ross §13 and §21, Folland §1.6)
- Connectedness (Ross §22, Folland §1.7)
- Uniform Continuity (Ross §21, Folland §1.7)
- Differentiability (Folland §2.1-2.2)
- The chain rule (Folland §2.3)
- Vector valued functions and there derivatives (Folland §2.10)

Example questions:

1. Complete the following definitions:
   (a) Let $S \subset \mathbb{R}^n$. Then $\mathbf{x} \in \mathbb{R}^n$ is a *boundary point* of S if
   (b) A set $S \subset \mathbb{R}^n$ is *arcwise connected* if
   (c) A sequence $\{\mathbf{x}_k\}$ in $\mathbb{R}^n$ *converges to a limit* $\mathbf{L} \in \mathbb{R}^n$ if
2. Complete the following definitions:
   (a) Let (S, d) be a metric space. A subset $K \subset S$ is *compact* if ...
   (b) A subset $E \subset \mathbb{R}^n$ is *path connected (or arcwise connected)* if ...
   (c) Let $f : \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{u} \in \mathbb{R}^n$ be a unit vector. The *directional derivative* of f at $\mathbf{a} \in \mathbb{R}^n$ in the direction $\mathbf{u}$ is ...
3. Let (S, d) be a metric space. Suppose $\{s_n \in S\}$ be a sequence converging to $s \in S$. Suppose $\{t_n \in S\}$ is a sequence of points so that $d(s_n, t_n) < \frac{1}{n}$ for all n. Prove that $\{t_n\}$ converges to s.
4. Let $f(x,y) = \begin{cases} x\cos(\frac{1}{y}) & \text{if } y \neq 0 \\ 0 & \text{if } y = 0. \end{cases}$
   (a) Show that f is continuous at $(0,0)$.
   (b) Is f differentiable at $(0,0)$? Why or why not?
5. Find counterexamples to the following false statements.
   (a) Any continuous image of an open ball in $\mathbb{R}^2$ is bounded.

(b) If $f : \mathbb{R} \to \mathbb{R}^2$ is differentiable, then so is the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = \max\big(f_1(x), f_2(x)\big)$.

(c) If $S_k \subset \mathbb{R}^2$ is a sequence of bounded non-empty open sets, and $S_{k+1} \subset S_k$ for all k, then the intersection $\bigcap_{k=1}^{\infty} S_k$ is non-empty.

6. Let $S = \{(x, y, z) \in \mathbb{R}^3 \;:\; x^2 + y^2 + z^2 = 1\}$. Let $\mathbf{g} : \mathbb{R} \to \mathbb{R}^3$ be a differentiable function so that $\mathbf{g}(t) \in S$ for all $t \in \mathbb{R}$. Use the chain rule to prove that $\mathbf{g}(t) \cdot \mathbf{g}'(t) = 0$ for all $t \in \mathbb{R}$. (It may be useful to consider $\mathbf{g}(t) = \big(g_1(t), g_2(t), g_3(t)\big)$.)

7. Recall a function $f : X \to Y$ is *surjective* (or *onto*) if for all $y \in Y$ there is an $x \in X$ so that $f(x) = y$.

   Let $B = \{(x, y) \in \mathbb{R}^2 \;:\; x^2 + y^2 < 1\}$. Prove that there is no continous surjective map $\mathbf{f} : \overline{B} \to B$.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. The *graph of* f is the set

$$G = \big\{(x, y) \in \mathbb{R}^2 \;:\; y = f(x)\big\}.$$

   Prove that G is closed.

9. Let $g(a, b, c)$ be a C^2 function $\mathbb{R}^3 \to \mathbb{R}$. Let $f(x, y) = g(x^2, xy, y^2)$.

   (a) Compute $\frac{\partial f}{\partial x}(x, y)$ in terms of x, y and partial derivatives of g.

   (b) Compute $\frac{\partial^2 f}{\partial y \partial x}(x, y)$ in terms of x, y and partial derivatives of g.

10. Let $S = \{(x, y) \in \mathbb{R}^2 \;:\; (x, y) \neq \mathbf{0}\}$. Prove that any continuous function $f : S \to \mathbb{R}$ satisfying the equation

$$f(-x, -y) = -f(x, y) \quad \text{for all } (x, y) \in S$$

    has a zero. That is, prove that there is a point $(x_0, y_0) \in S$ with $f(x_0, y_0) = 0$.

11. (a) Complete the following definition. Let $S \subset \mathbb{R}^n$ be an open set. The function $f : S \to \mathbb{R}$ is differentiable at $\mathbf{a} \in S$ if ...

    (b) Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies $f(0, 0) = 0$ and

$$x - x^2 - y^2 \le f(x, y) \le x + x^2 + y^2 \quad \text{for all } (x, y) \in \mathbb{R}^2.$$

    Prove that f is differentiable at $(0, 0)$.