Periodicity

January 30, 2019

1 Periodicity

We consider the doubling map on the circle $\mathbb{R}/\mathbb{Z}$:

```
In [1]: D(x) = 2*x - floor(2*x)
        show(D)
```

```
x |--> 2*x - floor(2*x)
```

Some example applications:

```
In [2]: D(1/4)
```

```
Out[2]: 1/2
```

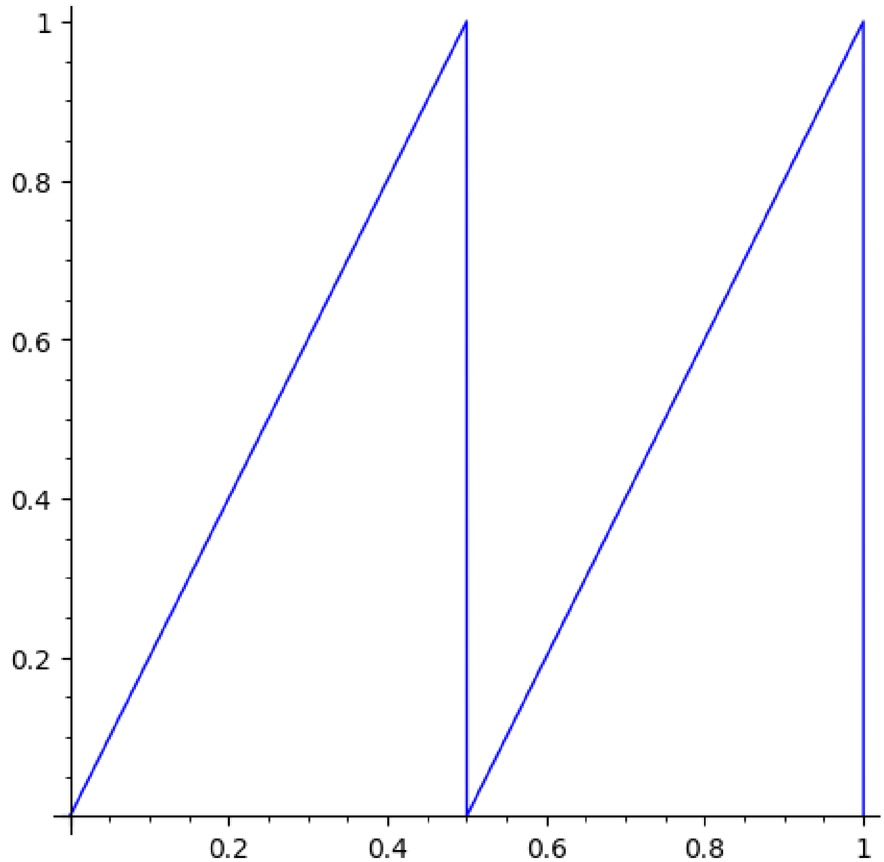
```
In [3]: D(2/3)
```

```
Out[3]: 1/3
```

Here we plot the function over the interval (0,1):

```
In [4]: plot(D,(x,0,1), aspect_ratio=1)
```

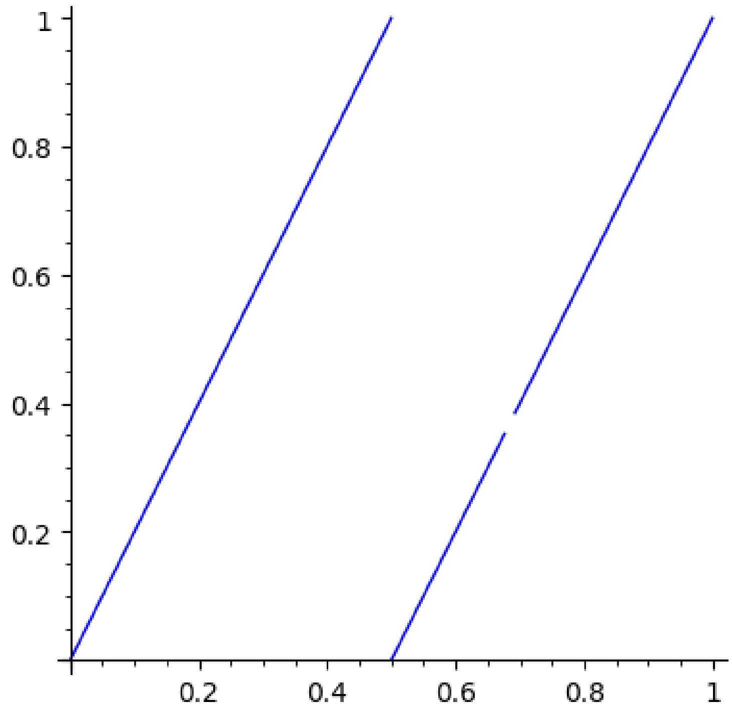
```
Out[4]:
```



As a remark, we can get rid of those vertical lines (which arise because Sage doesn't realize that the function is not continuous) using the exclude parameter:

```
In [5]: plot(D,(x,0,1), aspect_ratio=1, exclude=[1/2,1])
        # exclude corrects the plotting for discontinuities
```

```
Out[5]:
```




```
In [6]: def forward_orbit(x, T, N):
            '''
            Return the list [x, T(x), ..., T^N(x)].
            '''
            orbit = [x] # Start of the orbit.
            y = x
            for i in range(N):
                y = T(y) # Redefine y to be T(y)
                orbit.append(y) # Add y at the end of the orbit.
            return orbit
```

Recall that a fixed point of D is a value x so that $D(x) = x$. Zero is a fixed point:

```
In [7]: D(0)

Out[7]: 0
```

A periodic point of D is a point x so that $D^k(p) = p$ for some $k \geq 1$. The least period of p is the smallest such k. We say x has period k if $D^k(x) = x$.

The number $\frac{1}{3}$ has least period two, since the following output shows that $D(1/3) = 2/3$ and $D^2(1/3) = 1/3$.

```
In [8]: forward_orbit(1/3, D, 2)

Out[8]: [1/3, 2/3, 1/3]
```

A cobweb plot is a useful way to visualize an orbit of a map $T : \mathbb{R} \to \mathbb{R}$. It involves several things: * The graph of the function f. * The diagonal (the graph of the identity map) * The orbit. The orbit is visualized as the sequence of points (the cobweb path)

$$[(x,x),(x,T(x)),(T(x),T(x)),(T(x),T^2(x)),\ldots].$$

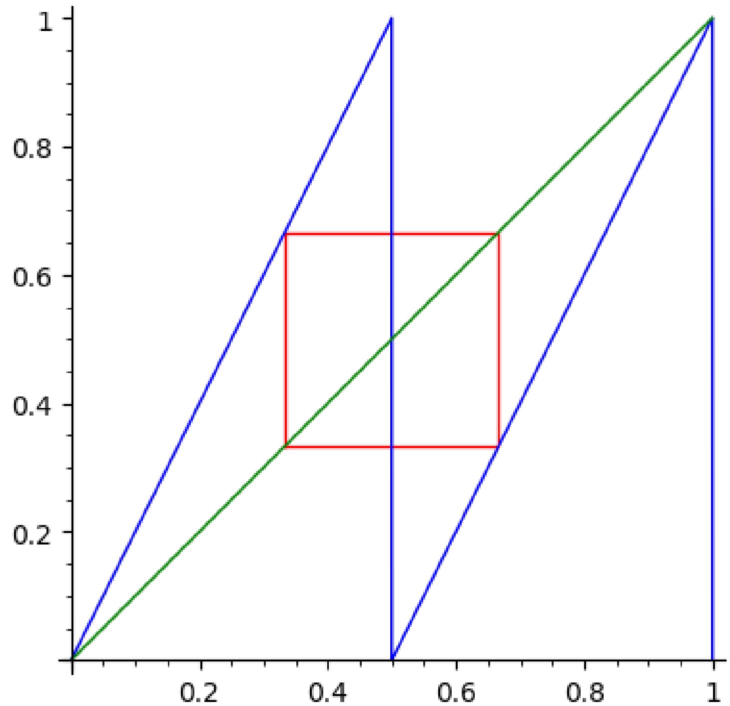
The following function draws a cobweb plot of the orbit of x, connecting (x,x) to $\big(T^N(x),T^N(x)\big)$ by a cobweb path:

```
In [9]: def cobweb(x, T, N, xmin, xmax):
            cobweb_path = [(x,x)]
            for i in range(N):
                y = T(x) # Reassign y to be T(x).
                cobweb_path.append( (x,y) )
                cobweb_path.append( (y,y) )
                x = y # Reassign x to be identical to y.
            cobweb_plot = line2d(cobweb_path, color="red", aspect_ratio=1)

            function_graph = plot(T, (xmin, xmax), color="blue")

            # define the identity map:
            identity(t) = t
            id_graph = plot(identity, (xmin, xmax), color="green")

            return cobweb_plot + function_graph + id_graph
```

Here is the cobweb plot of 1/3:

```
In [10]: plt = cobweb(1/3, D, 2, 0, 1)
         plt
```

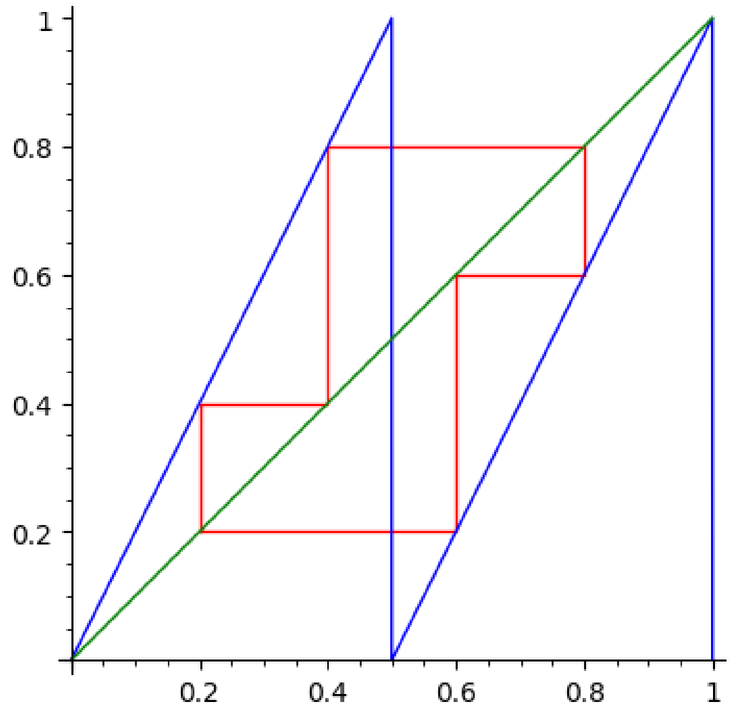
Out[10]:



The point 1/5 has period 4:

```
In [11]: plt = cobweb(1/5, D, 4, 0, 1)
         plt
```

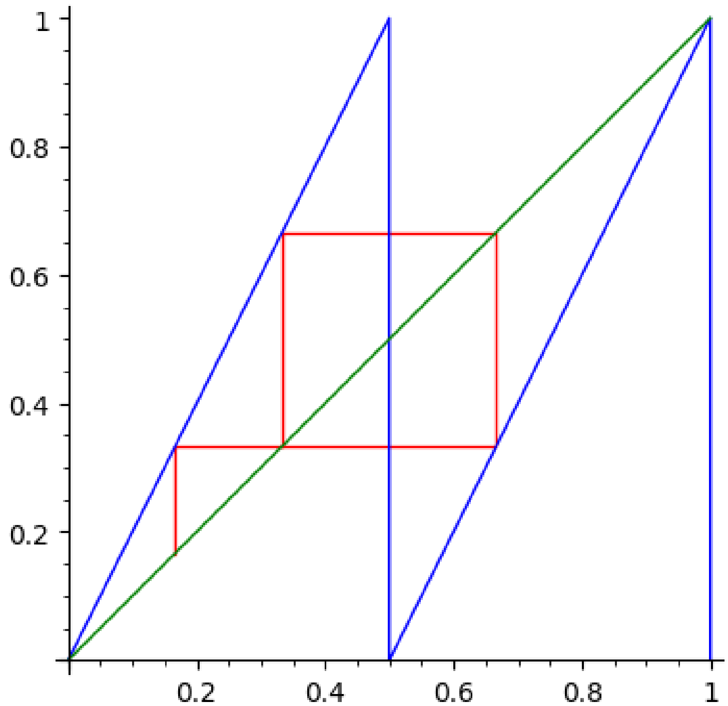
Out[11]:



Here is another phenomenon. The point $1/6$ is pre-periodic or eventually periodic. This means that there is a $k > 0$ so that $D^k(1/6)$ is periodic. For $1/6$, this k is one since $D(1/6) = 1/3$, and above we showed that $1/3$ is period 2:

In [12]:
```
plt = cobweb(1/6, D, 3, 0, 1)
plt
```

Out[12]:



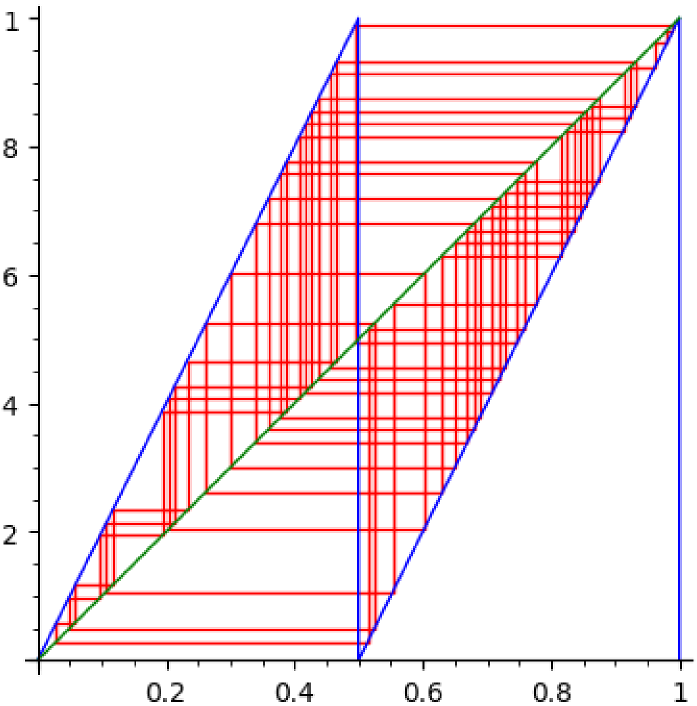
Lets compute the least period of 73/103:

In [13]:
```
x = D(73/103)
k = 1
while x != 73/103:
    x = D(x)
    k = k+1
print( "73/103 has least period " + str(k) )
```

```
73/103 has least period 51
```

In [14]:
```
cobweb(73/103, D, 51, 0, 1)
```

Out[14]:



Exercise: Think about why if p/q is a fraction, then it must be either periodic or eventually periodic under D. Under what conditions is it periodic? When is it eventually periodic?