Homeomorphisms_of_R

February 6, 2019

1 Homeomorphisms of $\mathbb{R}$

1.1 Orientation-preserving homeomorphisms

A homeomorphism of a topological space X is a continuous map $h : X \to X$ which has a continuous inverse $h^{-1} : X \to X$.

In advanced calculus, you should have learned that a continuous map $h : \mathbb{R} \to \mathbb{R}$ is a homeomorphism if and only if it is one-to-one and onto. (This does not hold for all spaces!)

A homeomorphism $h : \mathbb{R} \to \mathbb{R}$ is orientation-preserving if $x < y$ implies $h(x) < h(y)$.

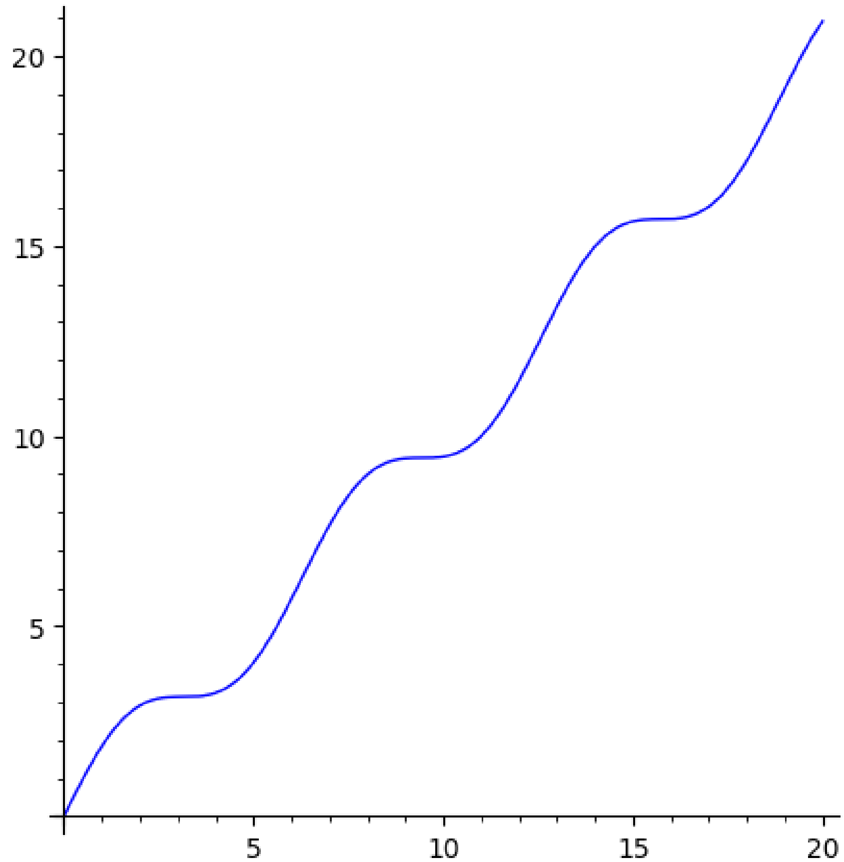
Since the following function has derivative which is non-negative and not identically zero on any interval, it is a homeomorphism of $\mathbb{R}$.

```
In [1]: f(x) = x + sin(x)
```

Here we plot the function over the interval (0,10):

```
In [2]: plot(f,(x,0,20), aspect_ratio=1)
```

```
Out[2]:
```




```
In [3]: def forward_orbit(x, T, N):
            '''
            Return the list [x, T(x), ..., T^N(x)].
            '''
            orbit = [x] # Start of the orbit.
            y = x
            for i in range(N):
                y = T(y) # Redefine y to be T(y)
                orbit.append(y) # Add y at the end of the orbit.
            return orbit
```

A cobweb plot is a useful way to visualize an orbit of a map $T : \mathbb{R} \to \mathbb{R}$. It involves several things: * The graph of the function f. * The diagonal (the graph of the identity map) * The orbit. The orbit is visualized as the sequence of points

$$[(x,x),(x,T(x)),(T(x),T(x)),(T(x),T^2(x)),\ldots].$$

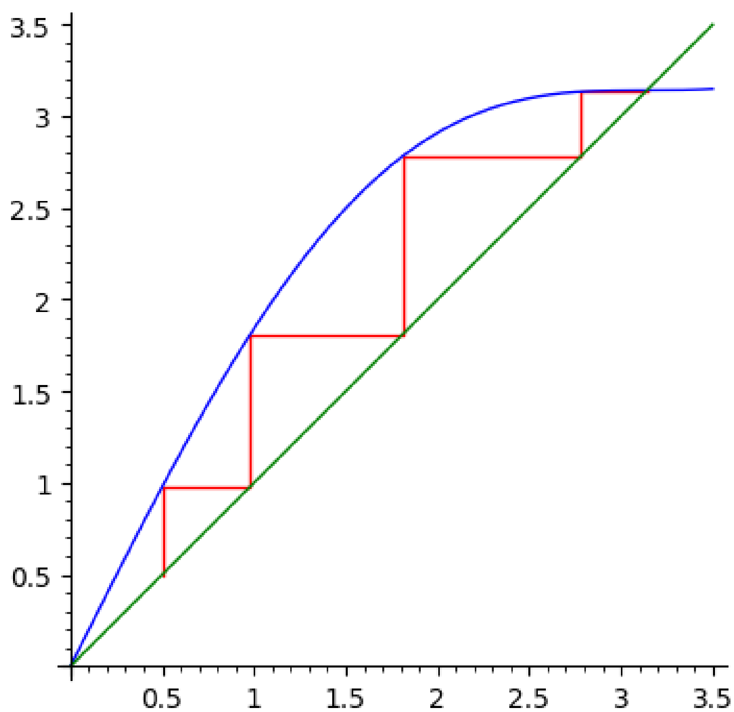
Here we define the identity map:

```
In [4]: identity(x) = x
```

In [5]:
```
def cobweb(x, T, N, xmin, xmax):
    cobweb_path = [(x,x)]
    for i in range(N):
        y = T(x) # Reassign y to be T(x).
        cobweb_path.append( (x,y) )
        cobweb_path.append( (y,y) )
        x = y # Reassign x to be identical to y.
    cobweb_plot = line2d(cobweb_path, color="red", aspect_ratio=1)

    function_graph = plot(T, (xmin, xmax), color="blue")

    # define the identity map:
    identity(t) = t
    id_graph = plot(identity, (xmin, xmax), color="green")

    return cobweb_plot + function_graph + id_graph
```

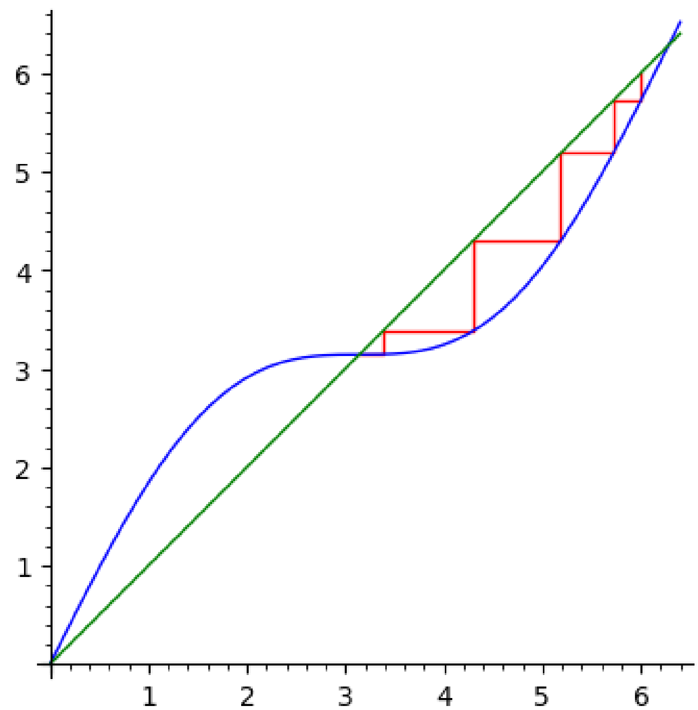
In [6]:
```
plt = cobweb(0.5, f, 10, 0, 3.5)
plt
```

Out[6]:



In [7]:
```
plt = cobweb(6, f, 10, 0, 6.4)
plt
```

Out[7]:



The stable set of a point periodic point p is the set of points x so that

$$\lim_{n\to\infty} dist\left(T^n(p), T^n(x)\right) = 0.$$

The set $W^s(p)$ denotes the stable set of p. We can see from the above cobweb plots that for $T(x) = x + sin(x)$, we have that $W^s(\pi)$ is the open interval $(0, 2\pi)$.

If $x \in W^s(p)$ we say x is forward asymptotic to p.

We remark that if $T : \mathbb{R} \to \mathbb{R}$ is an orientation-preserving homeomorphism, we can continuously extend the definition of T so that $T(+\infty) = +\infty$ and $T(-\infty) = -\infty$.

The following theorem completely describes the longterm behavior of orienation preserving homeomorphisms:

Theorem.

- Let $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{+\infty\}$ be fixed points with $a < b$ such that for every $x \in (a, b)$, $T(x) > x$. Then $(a, b) \subset W^s(b)$.
- Let $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{+\infty\}$ be fixed points with $a < b$ such that for every $x \in (a, b)$, $T(x) < x$. Then $(a, b) \subset W^s(a)$.

1.2 Orientation reversing homeomorphisms

A homeomorphism $T : \mathbb{R} \to \mathbb{R}$ is orientation-reversing if $x < y$ implies $T(x) > T(y)$.

If $T : \mathbb{R} \to \mathbb{R}$ is an orientation reversing homeomorphism then T extends so that $T(+\infty) = -\infty$ and $T(-\infty) = +\infty$.

We have the following result:

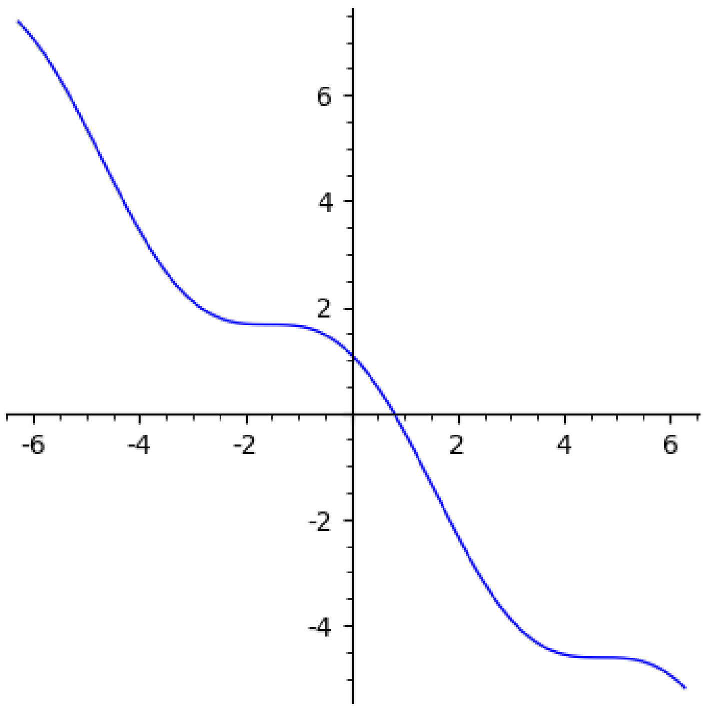
Proposition. If $T : \mathbb{R} \to \mathbb{R}$ is an orientation-reversing homeomorphism, then T has a unique fixed point in $\mathbb{R}$.

Existence of a fixed point is a consequence of the Intermediate Value Theorem. Uniqueness is a consequence of the definition of orientation-reversing.

Consider the following example:

In [8]:
```
f(x) = cos(x) - x + 1/10
plot(f,-2*pi,2*pi, aspect_ratio=1)
```

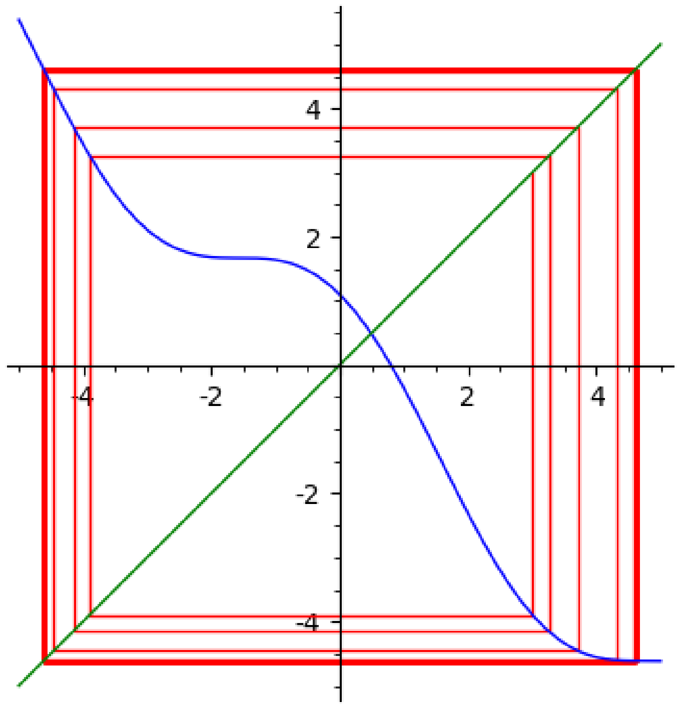
Out[8]:



The following is the cobweb plot of the orbit of 3.

In [9]:
```
cobweb(3.0, f, 20, -5, 5)
```

Out[9]:



It seems to be approaching a period two orbit, which is further supported by the following:

```
In [10]: forward_orbit(3.0, f, 20)

Out[10]: [3.00000000000000,
          -3.88999249660045,
          3.25721383591138,
          -4.15053714995653,
          3.71778289092915,
          -4.45632730660125,
          4.30305469756698,
          -4.60105339417880,
          4.58994767759334,
          -4.61208327222891,
          4.61194567936355,
          -4.61222017254902,
          4.61221879282579,
          -4.61222154535601,
          4.61222153155839,
          -4.61222155908446,
          4.61222155894648,
          -4.61222155922174,
          4.61222155922036,
          -4.61222155922312,
          4.61222155922310]
```

Also observe that:

Proposition. If $T : \mathbb{R} \to \mathbb{R}$ is an orientation-reversing homeomorphism, then $T^2 : \mathbb{R} \to \mathbb{R}$ is an orientation-preserving homeomorphism.

```
In [11]: g(x) = f(f(x))
```

```
In [12]: plot(g, -5, 5, aspect_ratio=1) + plot(x, (x,-5, 5), color="red")
```

Out[12]: