Bifurcations in homeomorphisms of R

February 7, 2019

1 Bifurcations in homeomorphisms of $\mathbb{R}$.

We will consider the family of maps

$$f_c(x) = \frac{1}{2}(e^x + x - c).$$

We can see this is a homeomorphism of $\mathbb{R}$ because the derivative is everywhere in the interval $[\frac{1}{2}, +\infty)$. Another nice property of the map is that $f_c'(0) = 1$ for all c. Also f' is increasing so that this is the only point where f_c' is zero.

```
In [1]: # This function returns the map f_c.
        def f(c):
            m(x) = 1/2*(e^x + x - c)
            return m
```

```
In [2]: f_1 = f(1)
        x = var("x")
        f_1(x)
```

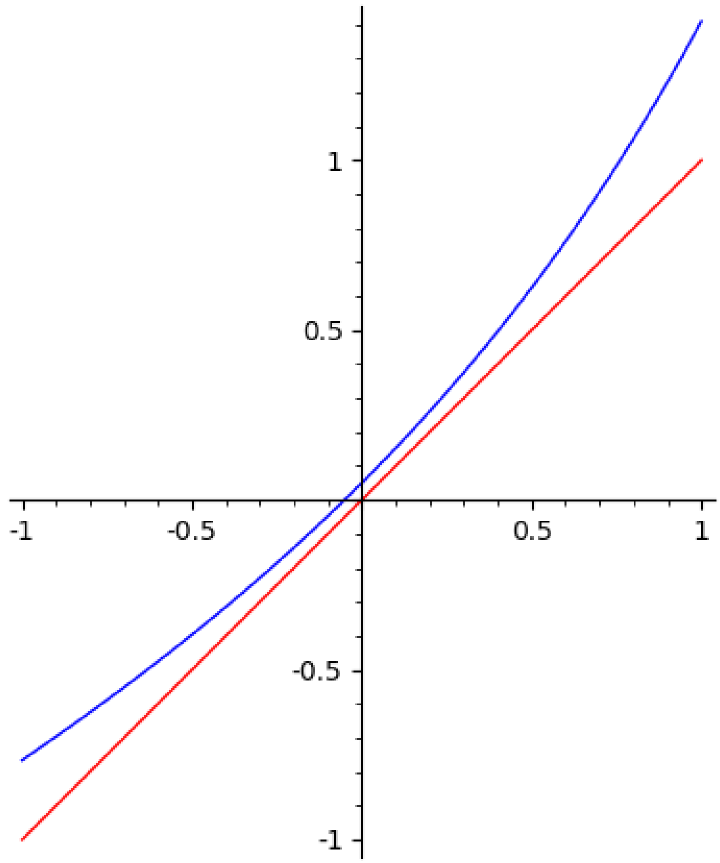
```
Out[2]: 1/2*x + 1/2*e^x - 1/2
```

```
In [3]: # The identity map
        identity(x) = x
```

A bifurcation occurs at the value $c = 1$. Here we plot some nearby values

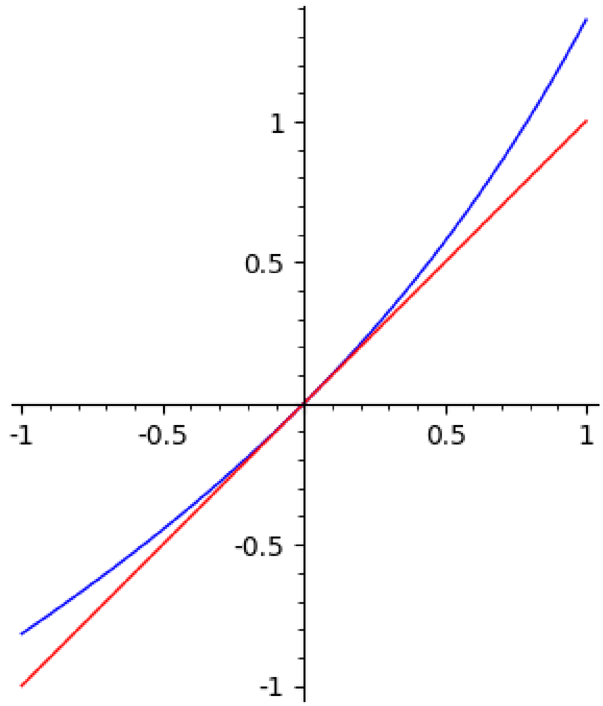
```
In [4]: # Plot of f_0.9
        plot(f(0.9),-1,1,aspect_ratio=1)+plot(identity, color="red")
```

```
Out[4]:
```



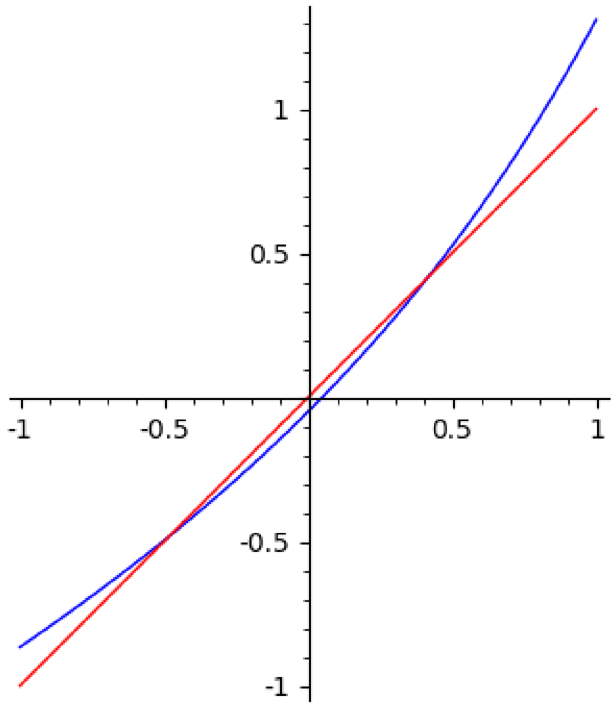

```
In [5]: # Plot of f_1
        plot(f(1),-1,1,aspect_ratio=1)+plot(identity, color="red")
```

Out[5]:




```
In [6]: # Plot of f_1.1
        plot(f(1.1),-1,1,aspect_ratio=1)+plot(identity, color="red")
```

Out[6]:



A bifurcation is a sudden change in the dynamics as we change the parameters of a family of dynamical systems. In this case, a bifurcation occurs at the value $c = 1$: * For values of $c < 1$: For every $x \in \mathbb{R}$, $\lim_{n \to +\infty} f_c^n(x) = +\infty$. That is, $W^s(+\infty) = \mathbb{R}$. * At the value $c = 1$: The map f_1 has a single fixed point, $f_1(0) = 0$. For values of $x < 0$, we have $\lim_{n \to +\infty} f_1^n(x) = 0$. For values of $x > 0$, we have $\lim_{n \to +\infty} f_1^n(x) = +\infty$. That is,

$$W^s(0) = (-\infty, 0] \quad \text{and} \quad W^s(+\infty) = (0, +\infty).$$

* At values of $c > 1$: The map f_c has two fixed points, denote them by a and b with $a < b$. The point a is an attracting fixed point while b is repelling. We have

$$W^s(a) = (-\infty, b) \quad \text{and} \quad W^s(+\infty) = (b, +\infty).$$

1.1 Visualizing the maps through a vector field.

We can visualize this bifurcation in the (x, c) plane, where dynamics in the horizontal line of height c represent the action of f_c. First, let us compute the fixed points.

Observe that the x value of a fixed point uniquely determines the c value:

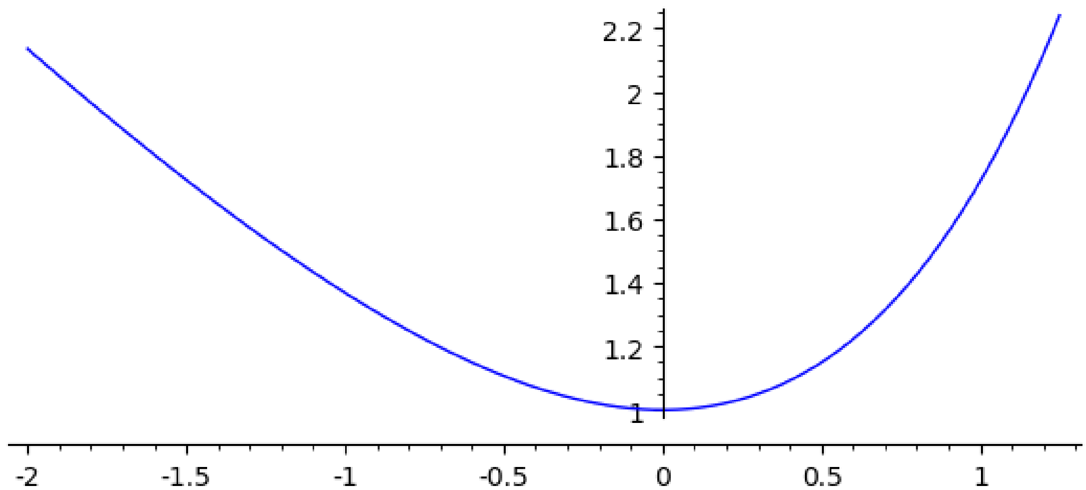
```
In [7]: c=var("c")
        x=var("x")
        solve(f(c)(x)==x,c)
```

```
Out[7]: [c == -x + e^x]
```

```
In [8]: c_value_of_fixed_point(x) = e^x - x
```

```
In [9]: fixed_point_plot = plot(c_value_of_fixed_point, -2, 1.25, aspect_ratio=1)
        fixed_point_plot
```

Out[9]:



Since c is a parameter, it is constant under iteration. We define the map

$$F(x,c) = \big(f_c(x), c\big).$$

```
In [10]: F(x,c) = (f(c)(x), c)
         F(x,c)
```

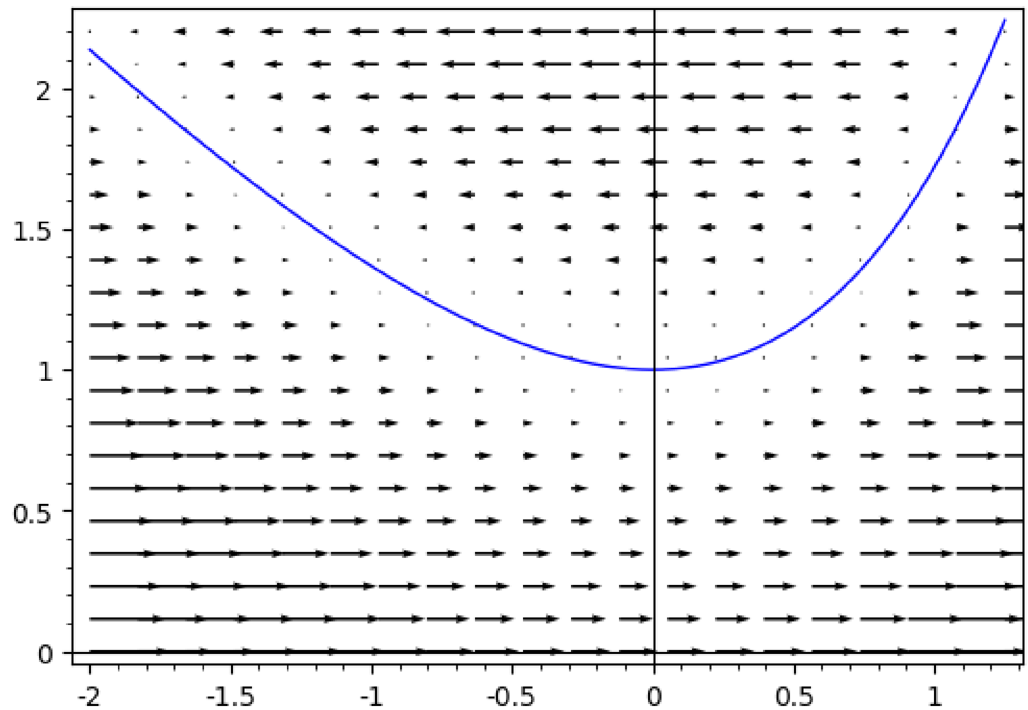
```
Out[10]: (-1/2*c + 1/2*x + 1/2*e^x, c)
```

We can visualize F as a vector field. At each point (x,c), we join (x,c) to its image $F(x,c)$ by a displacement vector with value $F(x,c) - (x,c)$. We just compute this to be:

```
In [11]: V(x,c) = (-1/2*c + 1/2*x + 1/2*e^x - x, 0)
```

```
In [12]: fixed_point_plot + plot_vector_field(V(x,c), (x,-2,1.25), (c,0,2.2))
```

Out[12]:


2
1.5
1
0.5
0
-2
-1.5
-1
-0.5
0
0.5
1