Math A4500: Midterm 2 Study Guide

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Disclaimer. This test is just a recommendation of things to study. You may be asked about things that do not appear here. I recommend ensuring that you can do all the problems in the sections covered in the book.

Definitions. You will be asked to define several terms on the test. The following is a list of terms which might appear.

Cantor set, totally disconnected, perfect, sequence space (or shift space), shift map, itinerary (or symbolic code), topological conjugacy, topologically transitive, sensitive dependence on initial conditions, chaotic, topologically semi-conjugate, C^r*-distance,* C^r*-structurally stable*

Theorems. Theorems and results from the book you should be able to prove.

- Proposition 6.5: The shift map is continuous.
- Proposition 6.6: Dynamical facts about the shift map on Σ_2.
- Theorem 7.2, 7.3, 7.5: The coding map is a topological conjugacy and consequences.
- Example 8.9: Example of Semiconjugacy, F_4 and consequences.
- Theorem 9.8: The local behavior of a map near a hyperbolic fixed point is locally conjugate to its derivative.
- Theorem 10.1: Period 3 implies all other periods.

Techniques. You should be able to use the various techniques we have developed for understanding particular dynamical systems acting on subsets of the real line. For instance, you should be able to:

- Prove that the shift map is continuous, and that a shift space is a Cantor set.
- Prove dynamical results about a Shift Space. See Proposition 6.6.
- Prove that the dynamics of f on a set Λ is topologically conjugate to a shift map on a shift space.
- Prove that a system is chaotic.
- Prove two systems are topologically conjugate using fundamental domains. This was done in class, but also see example 9.4.
- Use topological conjugacy to prove dynamical statements about a map. We discussed in class how f and g are topologically conjugate then they have similar dynamical properties. See Theorem 7.5, for instance.

- Prove a system is C^r-structurally stable, or not C^r structurally stable for a fixed r and a fixed map f.
- Sharkovsky's theorem: You should be able to use the techniques from §1.10 to find periodic orbits for a continuous map $f : \mathbb{R} \to \mathbb{R}$.

Types of problems. There will be between four and six multi-part problems on the midterm. You'll have the full class period (100 minutes) to complete the midterm. Problems of the following forms are likely to appear:

- *Homework*: Problems similar to assigned homework problems.
- *Recall something, then prove something*: Problems that as you to recall a definition or a major result, and then use it in a proof.
- *Math comprehension:* The problem states a new definition, which may not have seen before, and asks you to use the definition in basic proofs.
- *Prove a result:* Problems which ask you to prove a result which is proved in the book. See the section titled "Theorems" above.

Practice problems. These are some problems I gave in a midterm last time I taught this course.

1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and has the following orbit of period four:

$$f(0) = 2, \quad f^2(0) = 3, \quad f^3(0) = 1, \quad f^4(0) = 0.$$

Define the the intervals $A = [0, 1]$, $B = [1, 2]$ and $C = [2, 3]$.

   (a) Prove that f has a point of least period 13 which stays within $A \cup B \cup C$. Find an explicit sequence of intervals visited by such an orbit.

   (b) Use Sharkovsky's theorem to prove that f has points of all possible periods. (*Remarks:* It is easier to use Sharkovsky's theorem then to prove this fact with the methods used to prove Sharkovsky's theorem, however the application of Sharkovsky's theorem is indirect. You can use the lemmas from the proof of Sharkovsky's theorem without proving the lemmas.)

2. (The Logisitic Map) Let $F(x) = \mu x(1 - x)$ with $\mu > 2 + \sqrt{5}$. Let

$$\Lambda = \{x \ : \ F_\mu^n(x) \in [0, 1] \text{ for all } n \geq 0\}.$$

Let $p_0 \in (0, \frac{1}{2})$ and $p_1 \in (\frac{1}{2}, 1)$ be the points so that $f(p_0) = f(p_1) = 1$. Define $I_0 = [0, p_0]$ and $I_1 = [p_1, 1]$. Then, $\Lambda \subset I_0 \cup I_1$, and $|F'(x)| > k$ for some $k > 1$ and all $x \in I_0 \cup I_1$.

   (a) Describe how to code the orbit of a point in Λ using a one-sided shift space. That is, describe the map $s : \Lambda \to \Sigma_2$.

(b) Prove that the coding map s is injective.

(c) Assuming that the coding map s is a topological conjugacy between $F|_\Lambda : \Lambda \to \Lambda$ and $\sigma : \Sigma_2 \to \Sigma_2$, prove that F has a dense set of periodic points. (You do not need to prove that σ has a dense set of periodic points.)

3. (Shift spaces) Let $\mathcal{A} = \{0, 1\}$. Define the one-sided shift space

$$\Sigma = \{(s_0 s_1 s_2 \dots) \ : \ s_j \in \mathcal{A} \text{ for all integers } j \geq 0\}.$$

The shift space becomes a metric space when equipped with either of the following distance functions:

$$d(\mathbf{s}, \mathbf{t}) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i} \quad \text{or} \quad d'(\mathbf{s}, \mathbf{t}) = \begin{cases} 2^{-n} & \text{if } \mathbf{s} \neq \mathbf{t} \text{ and } n = \min\{i : s_i \neq t_i\} \\ 0 & \text{iff } \mathbf{s} = \mathbf{t}. \end{cases}$$

Both d and d' determine the same topology. The shift map $\sigma : \Sigma \to \Sigma$ is defined by

$$\sigma(s_0 s_1 s_2 \dots) = (s_1 s_2 s_3 \dots).$$

Use the definitions above and either of the metrics to answer the following questions.

(a) Is the map σ a homeomorphism? Why or why not?

(b) Prove that σ-periodic points are dense in Σ.

(c) Define *topologically transitive.*

(d) Prove that the shift map is topologically transitive on Σ.

4. Provide short answers to the following questions. You do not need to prove your answer is correct.

(a) Let $\Sigma_2 = \{0, 1\}^{\mathbb{N}}$ be the one-sided shift on the alphabet $\{0, 1\}$, and let $\sigma : \Sigma_2 \to \Sigma_2$ be the shift map. Let $s = (0\ 0\ 0\ \dots)$. Recall that t is in the stable set $W(s)$ if $\lim_{n\to\infty} d\big(\sigma^n(t), \sigma^n(s)\big) = 0$. Give a concrete description of $W(s)$.

(b) Explain why if $f : \mathbb{R} \to \mathbb{R}$ has exactly one periodic orbit of least period n and this orbit has a multiplier of -1, then f is not structurally stable.

(c) Suppose $f : \mathbb{R} \to \mathbb{R}$ has an attracting periodic orbit. Explain why f can not have a dense set of periodic points.

5. Let X and Y be closed and bounded subsets of $\mathbb{R}$. Let $f : X \to X$ and $g : Y \to Y$ be continuous maps.

(a) *Complete the following definition:* The maps $f : X \to X$ and $g : Y \to Y$ are *topologically conjugate* if ...

(b) Prove that if f and g are topologically conjugate and f has a dense orbit, then g has a dense orbit.

6. (Shift spaces) Let $\mathcal{A} = \{0, 1\}$. Define the one-sided shift space

$$\Sigma = \{(s_0 s_1 s_2 \ldots) \ : \ s_j \in \mathcal{A} \text{ for all integers } j \geq 0\}.$$

The shift space becomes a metric space when equipped with either of the following distance functions:

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Both d and d' determine the same topology. The shift map $\sigma : \Sigma \to \Sigma$ is defined by

$$\sigma(s_0 s_1 s_2 \ldots) = (s_1 s_2 s_3 \ldots).$$

Use the definitions above and either of the metrics to answer the following questions.

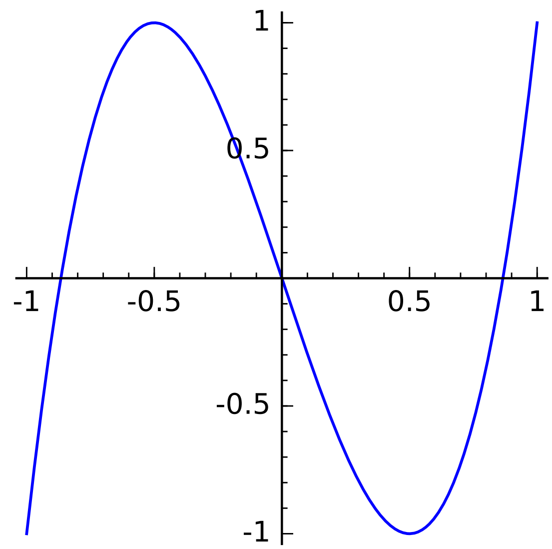
(a) Prove σ is continuous.

(b) Prove that Σ_2 is perfect, i.e., prove that for every $s \in \Sigma_2$, there is a sequence $t^n \in \Sigma_2$ converging to s with $t^n \neq s$ for all $n \in \mathbb{N}$.

7. Consider the function $G(x) = 4x(x^2 - 1) + x$. The graph of G is depicted below.

By computation it can be shown:

- $G([-1, 1]) = [-1, 1]$.
- The points -1, 0 and 1 are fixed.
- The critical points are $\pm\frac{1}{2}$.
- We have $G(\frac{1}{2}) = -1$ and $G(\frac{-1}{2}) = 1$.



(a) Use intervals and covering relations between intervals to prove that G has a periodic orbit with least period 3.

(b) Explain why G has orbits of all possible periods. (You can use any theorem you like.)

(c) Give a rigorous proof that zero is the only element of the set

$$\{x : \ G^n(x) \in [\tfrac{-1}{2}, \tfrac{1}{2}] \quad \text{for all integers } n \geq 0\}.$$

Hint: It may be useful to note that $G(\frac{1}{4}) = \frac{-11}{16}$, $G'(\frac{1}{4}) = \frac{-9}{4}$, $G(\frac{-1}{4}) = \frac{11}{16}$ and $G'(\frac{-1}{4}) = \frac{-9}{4}$.

8. (a) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a homeomorphism so that $f(x) > x$ for all $x \in \mathbb{R}$. Prove that $\lim_{n\to\infty} f^n(x) = +\infty$ for all $x \in \mathbb{R}$.

(b) Complete the following definition:
Let $f : A \to A$ and $g : B \to B$ be two maps with $A, B \subset \mathbb{R}$. We say f and g are *topologically conjugate* if ...

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be a homeomorphism so that $f(x) > x$ for all $x \in \mathbb{R}$ be as in the prior part, and let $g : \mathbb{R} \to \mathbb{R}$ be $g(x) = x+1$. Prove that f and g are topologically conjugate.
(*Hint:* Use part (a) and the similar fact that $\lim_{n\to\infty} f^{-n}(x) = -\infty$ for all $x \in \mathbb{R}$. This second fact can be assumed without proof.)

(d) Complete the following definition:
The C^r-*distance* between two C^r functions $f, g : \mathbb{R} \to \mathbb{R}$ is

(e) Complete the following definition:
A C^r function $g : \mathbb{R} \to \mathbb{R}$ is C^r-*structurally stable* if

(f) Prove that $g(x) = x + 1$ is C^1-structurally stable. (*Hint:* You can use part (c), even if you were not successful in writing a proof for that part.)

(g) Prove that $g(x) = x + 1$ is not C^0-structurally stable.