COMPARISON OF TOPOLOGIES ON $\mathbb{C}^X$

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For these notes, let E be a normed vector space. We'll use $|\cdot|$ to denote the norm on E.

For a topological space X, we'll be discussing topologies on E^X, the set of all functions from X to E. The goal is explain how the topologies are ordered in terms of strength.

The *topology of pointwise convergence*, denoted $\mathcal{T}_p$, is the weakest topology on E^X so that for each $x \in X$

$$eval_x : E^X \to E; \quad f \mapsto f(x)$$

is continuous. This means that the topology is generated by sets of the form $eval_x^{-1}(B)$ where $B \subset E$ is an open metric ball.

For $f \in E^X$, we'll define

$$\|f\| = \sup\{|f(x)| : \ x \in X\}.$$

This restricts to a norm on the collection of bounded functions in E^X, but $\|f\| = +\infty$ if f is unbounded. The *topology of uniform convergence* on E^X, denoted $\mathcal{T}_u$, is generated by sets of the form

$$B_\epsilon^u(f) = \{g \in E^X : \ \|g - f\| < \epsilon\}$$

where $f \in C^k$ and $\epsilon > 0$. (Here I'm using u in $\mathcal{T}_u$ and $B_\epsilon^u(f)$ to denote uniform.)

For $f \in E^X$ and $K \subset X$ compact, define

$$\|f\|_K = \sup\{|f(x)| : \ x \in K\}.$$

This function is a seminorm on E^X rather than a norm (unless $X = K$) because it can be that $\|f\| = 0$ even if f is not identically zero on X. For K as above, $\epsilon > 0$, and $f \in E^X$, we'll denote

$$B_\epsilon^K(f) = \{g \in E^X : \ \|g - f\|_K < \epsilon\}.$$

The *topology of uniform topology on compact sets*, denoted $\mathcal{T}_c$, is the topology generated by sets of the form $B_\epsilon^K(f)$.

Proposition 1. *The topology of pointwise convergence is weaker than the topology of uniform convergence on compact sets, i.e., that $\mathcal{T}_p \subset \mathcal{T}_c$.*

Proof. It suffices to show that sets of the form $eval_x^{-1}(B)$ are in $\mathcal{T}_c$ (since these sets generate $\mathcal{T}_p$). Fix x and B for the proof. Let $z \in E$ denote the center of B and ϵ denote the radius. Also fix K be the set of one element $\{x\}$, which is clearly compact.

Now let $f \in eval_x^{-1}(B)$ be arbitrary. Then $|f(x) - z| < \epsilon$. Let $r = \epsilon - |f(x) - z|$. Then we have $B_r^K(f) \subset eval_x^{-1}(B)$ since $g \in B_r^K(f)$ implies that $|g(x) - f(x)| < r$ and so

$$|g(x) - z| \leq |g(x) - f(x)| + |f(x) - z| < r + |f(x) - z| = \epsilon.$$

Date: October 13, 2019.

We have shown that $eval_x^{-1}(B)$, there is an $r = r(f)$ so that $f \in B_r^K(f) \subset eval_x^{-1}(B)$. The sets $B_r^K(f)$ are in $\mathcal{T}_c$ thus so is

$$eval_x^{-1}(B) = \bigcup_{f \in eval_x^{-1}(B)} B_{r(f)}^K(f).$$

□

Proposition 2. *The topology of pointwise convergence on compact sets is weaker than the topology of uniform convergence, i.e., that $\mathcal{T}_c \subset \mathcal{T}_u$.*

Proof. As in the previous proof, it suffices to prove that sets of the form $B_\epsilon^K(f)$ are open in $\mathcal{T}_u$. Fix $K \subset X$ compact, fix $f \in E^X$, and fix $\epsilon > 0$.

Now let $g \in B_\epsilon^K(f)$. Then $\|g - f\|_K < \epsilon$. Let $r = \epsilon - \|g - f\|_K$. We claim that $B_r^u(g) \subset B_\epsilon^K(f)$. To see this, let $h \in B_r^u(g)$. Then

$$\sup\{|h(x) - g(x)| : \ x \in K\} \leq \sup\{|h(x) - g(x)| : \ x \in X\} < r.$$

By the triangle inequality, for each $x \in K$ we have

$$|h(x) - f(x)| \leq |h(x) - g(x)| + |g(x) - f(x)|.$$

Thus,

$$\begin{aligned} \|h - f\|_K &= \sup\{|h(x) - f(x)| : \ x \in K\} \\ &\leq \sup\left\{\left(|h(x) - g(x)| + |g(x) - f(x)|\right) : \ x \in K\right\} \\ &\leq \sup\{|h(x) - g(x)| : \ x \in K\} + \sup\{|g(x) - f(x)| : \ x \in K\}. \\ &< r + \|g - f\|_K = \epsilon. \end{aligned}$$

□

Remark 3. *Letting $\mathcal{K} \subset \mathcal{P}(X)$ be the set of all compact sets, the topology of convergence on compact sets is generated by the sets of the form $B_\epsilon^K(f)$ for $K \in \mathcal{K}$, $\epsilon > 0$ and $f \in E^X$. If $\mathcal{K}' \subset \mathcal{P}(X)$ is any collection of subsets of X whose union is all of X, then the topology generated by sets of the form $B_\epsilon^{K'}(f)$ for $K' \in \mathcal{K}'$, $\epsilon > 0$ and $f \in E^X$ is stronger than the topology of pointwise convergence and weaker than the topology of uniform convergence.*

We are mostly interested in applying the topology of uniform convergence on compact sets to the space $C(X, E)$ of continuous functions from X into E. This space inherits the subspace (or relative) topology from $\mathcal{T}_c$ by viewing $C(X, E)$ as a subset of E^X. This is the *topology of uniform convergence on compact sets on* $C(X, E)$. By definition this topology is generated by sets of the form $B_\epsilon^K(f) \cap C(X, E)$ where again $K \subset X$ is compact, $\epsilon > 0$, and $f \in E^X$. But here f *need not be continuous.* It is useful to observe the following:

Proposition 4. *The topology of uniform convergence on compact sets on $C(X, E)$ is generated by sets of the form $B_\epsilon^K(g) \cap C(X, E)$ where $K \subset X$ is compact, $\epsilon > 0$, and $g \in C(X, E)$.*

Note also that for $g \in C(X, E)$, we have

$$B_\epsilon^K(g) \cap C(X, E) = \{h \in C(X, E) : \ |h(x) - g(x)| < \epsilon \text{ for all } x \in K\},$$

since continuous functions on compact sets attain their maxima.

Proof. It is enough to show that if $f \in E^X$ and $g \in B_\epsilon^K(f) \cap C(X, E)$, then there is an $r > 0$ so that $B_r^K(g) \subset B_\epsilon^K(f)$. As in the prior proofs, we set $r = \epsilon - \|f - g\|_K$, which is positive since $g \in B_\epsilon^K(f)$. Then by the triangle inequality (which holds for the seminorm $\|\cdot\|_K$), $h \in B_r^K(g)$ implies $\|h - g\|_K < r$ and thus

$$\|h - f\|_K \leq \|h - g\|_K + \|g - f\|_K < r + \|g - f\|_K = \epsilon,$$

which demonstrates that $h \in B_\epsilon^K(f)$. $\square$