Real Analysis Midterm Review
Fall 2019, Prof. Hooper

Disclaimer: This is just a list of things to study to prepare. You may be asked things not mentioned on this sheet and may need other concepts to do some problems. For example, this course assumes knowledge of calculus (for instance).

Topics covered:

- The real number system (Following Chapter 1 of Rudin)
- Cardinality
- Topological spaces (§4.1 of Folland)
- Metric Spaces (Chapter 2 of Rudin)
- Sequences (Chapter 3 of Rudin)
- Connected Sets (Chapter 2 of Rudin)
- Continuity (§4.2 of Folland)
- Nets (§4.3 of Folland, not explicitly tested but maybe useful to know)
- Compact sets (§4.4 of Folland, §2.3 of Rudin)
- Tychonoff's Theorem (§4.6 Folland)
- Locally compact Hausdorff spaces (§4.5 Folland)
- The Arzela-Ascoli Theorem (§4.6 Folland)
- The Stone-Weierstrass Theorem (§4.7 Folland)

What to study:

- Portions of Folland and Rudin mentioned above, including exercises (whether or not I assigned them).
- Definitions.
- Topics covered in class.
- Homework problems.
- Sample problems below.

Terms you should be able to define and use:
maximum, minimum, supremum, infinum, The completeness axiom, Cauchy sequence, complete metric space, same cardinality, smaller/greater cardinality, finite, countable, uncountable, seminorm, norm, triangle inequality, metric, metric space, bounded set in a metric space, topological space, open set, closed set, interior, boundary, closure, dense, first-countable space, second-countable space, neighborhood, separable space, accumulation point of a set, sequence, T_0 space, T_1 space, Hausdorff space, Regular space, Normal

space, subsequence, subsequential limit, cluster point, limit supremum, limit infimum, continuous function (definition for maps between topological spaces, as well as equivalent definitions for maps between metric spaces and for a linear map between normed vector spaces), homeomorphism, stronger/weaker topology, product topology, uniform norm, topology of pointwise convergence, topology of uniform convergence, bounded linear map, disconnected set, connected set, clopen, connected component, path connected, open cover, subcover, compact space, finite intersection property, sequentially compact, totally bounded, uniformly continuous, locally compact, support of a function, compactly supported function, vanishes at infinity, one point compactification, topology of uniform convergence on compact sets, σ-compact, pointwise totally bounded, equicontinuous, algebra, subalgebra

Results you should be able to state, use, and possibly prove: (This list is certainly not complete.)

- Archimedian property (prove)
- Denseness of $\mathbb{Q}$ in $\mathbb{R}$ (prove)
- Schröder-Bernstein Theorem (prove)
- Zorn's Lemma
- The power set of set always has strictly larger cardinality (prove)
- The real numbers are uncountable (prove)
- Metric spaces are normal (prove)
- In a Hausdorff spaces, limits of sequences/nets are unique (if they exists) (prove)
- Convergent sequences in a metric space are bounded and Cauchy (prove).
- The $B(X, \mathbf{F})$ of bounded functions $X \to \mathbf{F}$ (with $\mathbf{F} \in \{\mathbb{R}, \mathbb{C}\}$) is complete in the topology of uniform convergence. (prove)
- The uniform limit of continuous functions is continuous. (prove)
- Urysohn's lemma (for a normal space and for a locally compact Hausdorff space)
- Tietze Extension Theorem (for a normal space and for a locally compact Hausdorff space)
- Linear maps between normed vector spaces are continuous if and only if they are bounded. (prove)
- The continuous image of a compact set is compact. (prove)
- Compact Hausdorff spaces are normal. (prove)
- Closed graph theorem (for maps between topological spaces).
- Equivalent criteria for compactness of a metric space.
- Tychnoff's theorem.
- A continuous map from a compact metric space to a metric space is uniformly continuous.
- A normed vector space is locally compact if and only if it is finite dimensional. (Reisz)
- The collection of continuous functions that vanish at infinity is the closure of the collection of continuous compactly supported functions. (prove)

- In a locally compact metric space $C(X)$ is closed in the topology of uniform convergence on compact sets.
- Arzelà-Ascoli Theorem (both for compact space and for a σ-compact and locally compact Hausdorff space) (prove)
- Weierstrass Approximation Theorem
- Stone-Weierstrass Theorem (compact and locally compact versions)

Questions: The questions below appeared on the midterm when I taught the course last. The midterm had a total of six problems. (I left off one problem because it concerned a topic we haven't covered yet.)

1. Complete the following definitions:
   (a) Let X be a topological space, and let $A \subset X$. The *boundary of* A is...
   (b) A topological space X is *disconnected* if ...

2. Let $\{x_n\}$ be a sequence in a metric space X. Show that the set of limit points of convergent subsequences of $\{x_n\}$ is a closed subset of X.

3. A topological space is *separable* if it has a countable dense subset. A subspace is *second countable* if it has a countable base. Recall that a metric space is separable if and only if it is second countable.

   Show that any totally bounded metric space is separable. (*Remark:* If you can not recall the definition of totally bounded, then prove that every compact metric space is separable for a loss of a few points.)

4. Let X be a compact metric space and let $x_0 \in X$ be a basepoint. Let $C(X)$ denote the collection of continuous functions from X to $\mathbb{R}$ with the uniform norm. A function $f : X \to \mathbb{R}$ is said to be *K-Lipschitz* for some $K > 0$ if

   $$|f(x) - f(y)| \leq Kd(x, y) \quad \text{for each } x, y, \in X.$$

   Fix some $K > 0$. Show that the collection of functions

   $$\mathcal{L} = \{f \in C(X) \ : \ f \text{ is } K\text{-Lipschitz and } f(x_0) = 0\}$$

   is a compact subset of $C(X)$.

5. (a) State the Stone-Weierstrass Theorem. (It suffices to state a version for functions from a compact metric space into $\mathbb{R}$.)
   (b) Show that $\lim_{n\to\infty} \int_0^1 nx^n f(x)\, dx = f(1)$ for any continuous function $f : [0, 1] \to \mathbb{R}$.