Math 70100: Real Analysis(Prof. Hooper) **Due November 4, 2019**
Homework 7

1. *(Folland 4.7.68)* For a space X, let $C(X)$ denote the continuous functions from X to $\mathbb{R}$ equipped with the uniform norm. Let X and Y be compact Hausdorff spaces. Show that the algebra generated by functions of the form $f(x,y) = g(x)h(y)$, where $g \in C(X)$ and $h \in C(Y)$ is dense inside of $C(X \times Y)$.

2. *(Folland 4.7.69)* Let A be a nonempty set, and let $X = [0,1]^A$. Show that the algebra generated by the coordinate maps $\pi_a : X \to [0,1]$ and the constant function $\mathbf{1}$ is dense in $C(X)$.

3. *(Rephrased Lang III §4 # 19)* Let $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \ : \ x \geq 0\}$, and let $C_0(\mathbb{R}_{\geq 0})$ denote the continuous real-valued functions on $\mathbb{R}_{\geq 0}$ which vanish at infinity. Prove that $C_0(\mathbb{R}_{\geq 0})$ is the uniform closure of the collection of all functions of the form $e^{-x}p(x)$, where p is a polynomial. (*Lang's Hint; note he phrases this question differently:* First show that you can approximate e^{-2x} by $e^{-x}q(x)$ for some polynomial q(x), by using Taylor's formula with remainder. If p is a polynomial, approximate $e^{-nx}p(x)$ by $e^{-x}q(x)$ for some polynomial q.)

4. (Rudin §9.6) Define
$$f(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0), \\ \frac{xy}{x^2+y^2} & \text{otherwise.} \end{cases}$$
Prove that the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist everywhere, although f is not continuous at $(0,0)$.

5. (Rudin §9.7) Suppose that f is a real-valued function defined in an open set $E \subset \mathbb{R}^n$, and that the partial derivatives $D_1 f, \ldots, D_n f$ are bounded in E. Prove that f is continuous in E.

6. (Pugh §5.22) If Y is a metric space and $f : [a,b] \times Y \to \mathbb{R}$ is continuous, show that
$$F(y) = \int_a^b f(x,y)\ dx$$
is continuous.