Math 70100: Real Analysis(Prof. Hooper) **Due September 23, 2019**
Homework 3

1. *(Rudin 3.5)* For any two sequences of real numbers $\{a_n\}$ and $\{b_n\}$, prove that

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

provided the sum on the right is not of the form $\infty - \infty$.

2. *(Rudin 3.7)* Prove that the convergence of the series $\sum a_n$ implies the convergence of

$$\sum \frac{\sqrt{a_n}}{n}$$

if $a_n \geq 0$. (*Review basic facts about series if necessary before completing this problem.*)

3. *(Rudin 3.14; Cesàro summation)* If $\{s_n\}_{n=1}^{\infty}$ is a sequence of complex numbers, define its *arithmetic means* by

$$\sigma_n = \frac{s_1 + s_2 + \ldots + s_n}{n} \quad \text{for } n = 1, 2, 3, \ldots.$$

   (a) If $\lim s_n = s$, prove $\lim \sigma_n = s$.

   (b) Construct a sequence $\{s_n\}$ which does not converge, although $\lim \sigma_n = 0$.

   (c) Can it happen that $s_n > 0$ for all n and that $\limsup s_n = +\infty$, although $\lim \sigma_n = 0$?

   (*Remark:* Rudin has some additional parts to this problem related to partial converses to (a).)

4. *(Rudin 3.20)* Suppose $\{p_n\}$ is a Cauchy sequence in a metric space X, and some subsequence $\{p_n\}$ converges to a point $p \in X$, Prove that the full sequence $\{p_n\}$ converges to p.

5. *(Based on Pugh 2.92)* Let X be a topological space. Recall that a *neighborhood* N of $x \in X$ is a subset $N \subset X$ so that there is an open set $U \subset X$ with $x \in U$ and $U \subset N$. A *boundary point* of a set $A \subset X$ is a point $x \in X$ so that every neighborhood of x intersects both A and $X \smallsetminus A$. The boundary of A, ∂A, is the set of all boundary points of A.

   (a) Show that $\partial A = X \smallsetminus \big(\mathrm{Int}(A) \cup \mathrm{Int}(X \smallsetminus A)\big)$.

   (b) Explain why ∂A is closed.

   (c) Show that $\partial\partial A \subset \partial A$.

   (d) Show that $\partial\partial\partial A = \partial\partial A$.

   (e) Given an example which illustrates that $\partial\partial A$ may not equal ∂A.

6. *(Lang II.5.5c)* Let X be a metric space. For each $x \in X$, define the function f_x on X by $f_x(y) = d(x, y)$. Let $\|\cdot\|$ be the sup norm.

   (a) Show that $d(x, y) = \|f_x - f_y\|$.

   (b) Let a be a fixed element of X, and let $g_x = f_x - f_a$. Show that the map $x \mapsto g_x$ is a distance-preserving embedding of X into the normed space of bounded functions on X. (*Remark:* This shows that every metric space is isometric to a subset of a normed vector space.)

7. *(Modified from Lang II.5.3a)* Let ℓ^1 be the set of all sequences $\alpha = \{a_n\}_{n\in\mathbb{N}}$ of real numbers such that $\sum_{n\in\mathbb{N}} |a_n|$ converges. (If $\sum_{n\in\mathbb{N}} |a_n|$ converges, we say the series $\sum_{n\in\mathbb{N}} a_n$ *converges absolutely.*) Define

$$|\alpha| = \sum_{n\in\mathbb{N}} |a_n|.$$

   (a) Prove that $|\cdot|$ is a norm on ℓ^1.

   (b) Recall that a sequence $\{\alpha_n\}$ in a normed vector space is *Cauchy* if given any $\epsilon > 0$, there is an $N \in \mathbb{N}$ so that $|\alpha_m - \alpha_n| < \epsilon$ for $m, n \geq N$. A normed vector space is *complete* if all Cauchy sequences converge. Show that ℓ^1 is complete with the norm $|\cdot|$.