Math 70100: Real Analysis(Prof. Hooper) **Due December 6, 2019**
Homework 10

Remark: I am giving you extra time to complete this assignment because of Thanksgiving. I will probably assign one more homework and not collect it, so it would be good to finish before the due date.

1. (Folland §5.2 #20) Show that if M is a finite-dimensional subspace of a normed vector space X, there is a closed subspace N such that $M \cap N = \{0\}$ and $M + N = X$.

2. (Einseidler-Ward §7.1.3 # 7.11; Folland §5.2 # 23) Let $Y \subset X$ be a closed subspace of a normed vector space.
   (a) Show that $Y^{\perp} = \{x^* \in X^* | x^*(Y) = 0\}$ is a closed subspace.
   (b) Show that $(X/Y)^* = Y^{\perp}$ (that is, that there is a natural isometric isomorphism between the two).
   (c) Show that $Y^* = X^*/Y^{\perp}$.
   (d) Conclude that Y is reflexive if X is reflexive.

3. (Einseidler-Ward §7.1.3 # 7.11; Folland §5.2 # 25) Let X be a normed vector space and suppose that the dual X^* is separable. Show that X is also separable. In particular, if X is separable but X^* is not, then X cannot be reflexive. Find an example of a Banach space that is not reflexive for that reason.

4. (Folland §5.3 # 41) Let X be a vector space of countably infinite dimension (that is, every element is a finite linear combination of members of a countably infinite linearly independent set). There is no norm on X with respect to which X is complete. (Use the fact that a finite dimensional subspace of a normed vector space is homeomorphic to either $\mathbb{R}^n$ or $\mathbb{C}^n$ and is therefore complete and closed. Apply the Baire Category Theorem.)

5. (Spring 2015 Qual) Use the Baire Category Theorem to show that there exists a continuous function $f : [0, 1] \to \mathbb{R}$ which is not monotone on any interval of positive length.

6. (Spring 2011 Qual) Prove there is no sequence of continuous functions $f_n : \mathbb{R} \to \mathbb{R}_{>0}$ such that the sequence $\{f_n(x)\}$ is bounded for each $x \in \mathbb{Q}$ and unbounded for each $x \in \mathbb{R} \setminus \mathbb{Q}$.