Real Analysis Final Exam Review
Fall 2019, Prof. Hooper

What will be covered: Material from the whole semester will be covered. I have not written the final yet, but I anticipate about two thirds of the final to cover material taught after the first midterm and one third to cover material from before the midterm. I will likely have some multipart problems testing material from both before and after the midterm. **This review however concentrates more on material covered since the midterm. See the midterm review for a description of what was taught before the midterm.**

Disclaimer: This is just a list of things to study to prepare. You may be asked things not mentioned on this sheet and may need other concepts to do some problems. For example, this course assumes knowledge of calculus (for instance).

Topics covered: The following is a list of topics covered since the last midterm.

- Differentiation in $\mathbb{R}^n$ (followed Pugh, Chapter 5)
- Contraction Mapping Principle (Pugh Chapter 4, §5)
- The implicit and inverse function theorems
- Normed Vector Spaces (Folland §5.1)
- The sequence spaces ℓ^p for $p \in [1, \infty]$.
- The Hahn-Banach Theorem (Folland §5.2)
- The Baire Category Theorem (Folland §5.3 and additional applications)
- Hilbert spaces (Folland §5.5)
- Other topologies on normed vector spaces (end of Folland §5.4)

What to study:

- Portions of Folland and Pugh mentioned above, including exercises (whether or not I assigned them). I also like the treatment in Einseidler-Ward of some of these topics.
- Definitions.
- Topics covered in class.
- Homework problems.
- Sample problems below.
- Material discussed in the midterm review.

Terms you should be able to define and use:
differentiable, linear approximation, Fréchet (or total) derivative, partial derivative, chain rule, Leibniz rule (on derivatives of bilinear maps), second derivative, higher order derivatives, C^r ($r \in \{0, 1, 2, \ldots, \infty\}$), C^r-norm, contraction, fixed point, Lipschitz, C^r-diffeomorphism, norm, seminorm, normed vector space, norm

topology, equivalent norms, Banach space, absolutely convergent series, bounded linear map, operator norm, the space $L(X,Y)$, invertible, isomorphism, isometry, the sequence space ℓ^p and its norm, conjugate exponents ($\frac{1}{p}+\frac{1}{q}=1$), scalar product $\ell^p \times \ell^q \to \mathbb{C}$, linear functional, dual space, sublinear functional, reflexive space, completion (Folland pp. 159) , adjoint, nowhere dense, meager, residual, graph, inner product, pre-Hilbert space, Hilbert space, orthonormal set, Gram-Schmidt process, orthonormal basis, unitary map, $\ell^2(A)$ where A is a set, orthogonal projection, the weak topology on a normed vector space, weak topology on a dual space, the weak$*$ topology on a dual space, the strong operator topology on $L(X,Y)$, the weak operator topology on $L(X,Y)$.

Results you should be able to state, use, and possibly prove: (This list is certainly not complete.)

- Mean Value Theorem (1 variable version, Pugh Theorem 11 of Ch. 5, C^1 Mean Value Theorem (Pugh Thm. 12)
- Technique of passing derivative into an integral (Pugh Thm. 14 of Ch. 5) *(prove)*
- Symmetry of second derivative (Pugh Thm. 15 of Ch. 5), and higher derivatives (Pugh Cor. 18)
- Contraction mapping theorem *(prove)*
- Implicit Function Theorem
- Inverse Function Theorem
- A normed vector space is complete if and only if every absolutely convergent series converges. *(prove)*
- Equivalent conditions for continuity of a linear map (Folland Prop. 5.2) *(prove)*
- If Y is a Banach space, $L(X,Y)$ is complete. *(prove)*
- Young's inequality: If $\frac{1}{p}+\frac{1}{q}=1$, $a>0$, and $b>0$ then $ab \le \frac{a^p}{p}+\frac{b^q}{q}$.
- Hölder's Inequality: $|(x,y)| \le \|x\|_p \|y\|_q$.
- The map $\ell^p \to (\ell^q)^*$ (induced by the scalar product $\ell^p \times \ell^q \to \mathbb{C}$) is an isometry.
- If $p \in (1,\infty]$, then the map $\ell^p \to (\ell^q)^*$ is an isomorphism.
- For each $p \in [1,\infty]$, the space ℓ^p is complete.
- The Hahn-Banach Theorem (both Real and Complex variants in Folland §5.2) *(prove)*
- The Baire-Category Theorem (we discussed two versions)
- Open Mapping Theorem (Folland, Thm. 5.10) *(prove)*
- Closed Graph Theorem (Folland, Thm. 5.12) *(prove)*
- Uniform Boundedness Principle (Folland, Thm. 5.13; we discussed the version in Lang, Ch. 9, Thm 3.2) *(prove)*
- The CauchySchwarz inequality (Folland calls this the Schwarz inequality, see Thm 5.19)
- The Parallelogram Law and The Pythagorean Theorem.
- The relationship between a Hilbert space and its dual (Folland Thm 5.25; sometimes called one of the Riesz representation theorems) *(prove)*

- Bessel's inequality *(prove)*
- Every Hilbert space has an orthonormal basis (Folland Prop 5.28), and is therefore the same up to a unitary map to $\ell^2(A)$ (Folland 5.30)
- Alaoglu's Theorem. (Folland 5.18)

Related Qual Questions. These problems are related to the topics above and have appeared on past quals.

1. *(Fall 2008 qual)* Let H be a separable Hilbert space with inner product $(\cdot,\cdot)$ and $\{e_n\}_{n=1}^\infty$ be an orthonormal set in H, and let $x, y \in H$ be given by $x = \sum_{n=1}^\infty x_n e_n$ and $y = \sum_{n=1}^\infty y_n e_n$. Show that $(x, y) = \sum_{n=1}^\infty x_n y_n$.
2. *(Fall 2009 qual)* Let $\|\cdot\|$ be the sup norm on $C([0,1])$, the space of real-valued continuous functions on $[0,1]$. With this norm, $C([0,1])$ is a Banach space. Prove or disprove: This norm arises from an inner product on $C([0,1])$, i.e., $C([0,1])$ is a Hilbert space with this norm.
3. *(Fall 2014 qual)* Let f be a non-zero bounded linear functional on a normed space X. Find the distance between the origin and the hyperplane $\{x \in X : \ f(x) = 1\}$.
4. *(Fall 2015 qual)* Show that $[0,1]$ is not the disjoint union of countably many closed non-empty sets.
5. *(Spring 2015 qual)* Let T be a linear operator on a complex space X with inner product $\langle\cdot,\cdot\rangle$. If $\langle T(x), x\rangle = 0$ for all $x \in X$, prove that $T = 0$.
6. *(Spring 2016 qual)* Let V be a Hilbert space and let $T : V \to V$ be a linear operator satisfying $\|T\| \le 1$. Prove that $T - \sqrt{2}I$ is invertible or give a counterexample.
7. *(Spring 2016 qual)* Let H be an infinite-dimensional Hilbert space. Prove that
   (a) Any sequence $\{e_n\}_{n=1}^\infty$ of orthonormal elements converges weakly to 0.
   (b) Prove that any x in the unit ball $\{x : \ \|x\| \le 1\}$ is a weak limit of a sequence in the unit sphere $\{x : \ \|x\| = 1\}$, i.e., the unit sphere is dense in the unit ball in the weak topology.