Constructing the real numbers by completion

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A *Cauchy sequence* in $\mathbb{Q}$ is a sequence $\{x_n \in \mathbb{Q}\}_{n\in\mathbb{N}}$ so that for every $\epsilon \in \mathbb{Q}$ with $\epsilon > 0$ there is an integer N such that $n > N$ and $m > N$ implies $|x_n - x_m| < \epsilon$. We will let $\mathbf{C}$ denote the set of all Cauchy sequences of rational numbers.

1. Let $\{x_n\}$ and $\{y_n\}$ be in $\mathbf{C}$. Prove that their termwise sum $\{x_n + y_n\}_{n\in\mathbb{N}}$ and product $\{x_n y_n\}_{n\in\mathbb{N}}$ are Cauchy.

We say a sequence $\{x_n\}$ tends to zero (denoted $x_n \to 0$) if for every $\epsilon > 0$, there is an integer N so that $|x_n| < \epsilon$ whenever $n > N$. Let $\{x_n\}$ and $\{y_n\}$ be in $\mathbf{C}$. Write $\{x_n\} \sim \{y_n\}$ if their termwise difference (the sequence $\{x_n - y_n\}_{n\in\mathbb{N}}$) tends to zero.

2. Prove that $\sim$ is an equivalence relation on $\mathbf{C}$.

3. Prove that addition and multiplication are well defined on the quotient $\mathbf{C}/\sim$.

4. Note that a sequence $\{x_n\} \in \mathbf{C}$ tends to zero if and only if it is equivalent to the constant sequence $\{0\}_{n\in\mathbb{N}}$. Show that if $\{x_n\} \in \mathbf{C}$ is not equivalent to $\{0\}$ then the termwise multiplicative inverse $\{\frac{1}{x_n}\}_{n\in\mathbb{N}}$ is Cauchy. Show that multiplicative inversion is well defined on $\mathbf{C}/\sim$.

The Field Axioms are easy to check for $\mathbf{C}/\sim$. Also the rational numbers embed in $\mathbf{C}/\sim$ with each rational $\frac{p}{q}$ mapping to the equivalence class of the constant sequence.

We would also need to show that $\mathbf{C}/\sim$ is an ordered set. By unwinding definitions, note that $\{x_n\}$ and $\{y_n\}$ are equivalent if and only if for any $\epsilon > 0$, there is an N so that $n > N$ implies $|x_n - y_n| < \epsilon$. So, if $\{x_n\}$ and $\{y_n\}$ are not equivalent, there is an $\epsilon > 0$ so that for any N we can find an $n > N$ such that $y_n \geq x_n + \epsilon$ or $x_n \geq y_n + \epsilon$.

5. Say $\{x_n\}$ is bigger than $\{y_n\}$ if there is an $\epsilon \in \mathbb{Q}$ with $\epsilon > 0$ so that for any N we can find an $n > N$ so that $x_n \geq y_n + \epsilon$. Prove this relation is well defined on $\mathbf{C}/\sim$.

6. Prove that $>$ as defined on $\mathbf{C}/\sim$ is a order, i.e., that it is total and transitive.

7. Check that the two ordered field axioms hold for $\mathbf{C}/\sim$.

In an indication of faith, lets start writing $\mathbb{R}$ for $\mathbf{C}/\sim$. Then $\mathbb{Q}$ can both be considered a single number and the equivalence class of the corresponding constant sequence of rationals.

Now we will begin by working towards the proof of the Completeness Axiom (the least upper bound property).

8. Prove that $\mathbb{Q}$ is dense in $\mathbf{C}/\sim$ in the sense that $x, y \in \mathbb{R}$ and $x < y$ implies there is an $r \in \mathbb{Q}$ so that $x < r < y$.

9. Prove that $\mathbb{R}$ has the least upper bound property. (*Hint:* Give a set $S \subset \mathbb{R}$ with an upper bound, define a sequence $\{x_n\}$ according to the rule that

$$x_n = \min\left\{\frac{m}{n} : \ m \in \mathbb{Z} \text{ and } \frac{m}{n} \text{ is an upper bound for } S\right\}.$$

Show that $\{x_n\}$ is in $\mathbf{C}$ and thus determines a real number x. Show that x is the least upper bound for S.)