Math 323: Practice for the Final Exam

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Disclaimer. This test is just a recommendation of things to study and problems to work on. You may be asked about things that do not appear here. You should practice doing problems from the book in addition to the problems included in this sheet.

Covered Material. All unstarred sections in the book will be covered except sections 25 and 26. Material covered since the 3rd midterm will definitely appear.

Definitions to know: You need to be able to state and use the following definitions:

maximum, minimum, upper bound, lower bound, supremum, infimum, converge, diverge, limit, nonincreasing sequence, nondecreasing sequence, monotone sequence, Cauchy sequence, subsequence, subsequential limit, limit superior (lim sup), and limit inferior (lim inf), Series, Sequence of partial sums, Convergent series, Cauchy criterion, absolutely convergent, alternating series, continuous, bounded function, uniformly continuous, extension of a function, (two-sided) limit, right-handed limit (limit from the right), left-handed limit (limit from the left), power series, radius of convergence, interval of convergence, pointwise convergence of a sequence of functions, uniform convergence of a sequence of functions, differentiable, derivative, maximum, minimum, strictly increasing, strictly decreasing, increasing, decreasing, Taylor series, remainder $R_n(x)$, Taylor polynomial of degree n, partition, upper and lower Darboux sums, (Darboux) integrable, Darboux integral

Theorems. You should be able to state many of the named theorems in the book, and be able to use them in proofs. You should be able to prove many of the results in the proof, and in particular, be able to demonstrate a mastery of proving statements involving quantifiers (e.g., ϵ-δ proofs). Below is a partial list of such results. Bolded results are things I believe you should be able to prove.

4.4: Completeness Axiom, **4.6: Archimedian Property**, **4.7: Denseness of $\mathbb{Q}$**, **9.1: Convergent sequences are bounded**, **9.2-9.6: Algebraic properties of limits**, **10.2: Bounded monotone sequences converge**, *10.7: Relationship between limits, limit superior, and limit inferior,* **10.9: A convergent sequence is Cauchy**, **10.11: Sequences are convergent iff Cauchy**, *11.3: Every sequence has a monotone subsequence, 11.5: Bolzano-Weierstrass Theorem, 11.7-11.8 Theorems on the set of subsequential limits, 14.4: Cauchy Criterion,* **14.5: Limit associated to a converent series**, *14.6: Comparison test, 14.8: Ratio Test, 14.9: Root Test, 15.3: Alternating Series Theorem, 17.2: ϵ-δ definition of continuity, 17.3-17.5: Algebraic properties of continuity,* **18.1: Continuous functions attain extrema**, *18.2: Intermediate Value Theorem, 19.2: A continuous function on a closed interval is uniformly continuous,* **19.3: Uniformly continuous functions send Cauchy sequences to Cauchy sequences**, *19.5: Uniform continuity and extensions of continuous functions, 20.6: ϵ-δ definition of limits of functions, 23.1: Radius of convergence theorem, 24.3. The uniform limit of continuous functions is continuous,* 25.4: A uniformly Cauchy sequence of functions converges uniformly, 25.6: Cauchy criterion implies uniform convergence, 25.7: Weirstrass M-test, 26.1: Uniform convergence of power series, 26.4-26.5: Derivatives and integrals of power series, **28.2: Differentiability implies continuity**, *28.3: Algebraic properties of the derivative (***prove i-iii***), 29.1: Extremum implies zero derivative,* **29.2: Rolle's Theorem**, *29.3: The mean value theorem, 29.4: Derivative zero on an interval implies constant, 29.7: Derivative conditions for monotonicity. 29.8: Intermediate value theorem for derivatives, 29.9: Inverse functions and* *derivatives, 31.3 Taylor's Theorem, 32.2 and 32.3: Inequalities relating Darboux sums, 32.5 and 32.7: Characterizations of integrability,* **34.1: The fundamental theorem of calculus**, 34.2: Integration by parts, 34.3: The fundamental theorem of calculus, 34.4: Change of variables

Problems. What follows is a list of problems from former final exams.

1. True or false. No explanation is necessary.
   (a) If a function $f : \mathbb{R} \to \mathbb{R}$ is differentiable, then the derivative f' is always continuous.
   (b) All integrable functions $f : [0, 1] \to \mathbb{R}$ are continuous.
   (c) A composition of uniformly continuous functions is always uniformly continuous.
   (d) If $f : \mathbb{R} \to \mathbb{R}$ is continuous, and $(x_n \in \mathbb{R})$ is a Cauchy sequence, then $\big(f(x_n)\big)$ is always a Cauchy sequence.
   (e) If the series $\sum_{n=1}^{\infty} a_n$ converges, then the sequence (a_n) always also converges.
   (f) The set $\{x \in \mathbb{Q} : \ x \le a\}$ has a maximum whenever $a \in \mathbb{R}$.
   (g) If $S \subset \mathbb{R}$ and a sequence of functions $f_n : S \to \mathbb{R}$ converges uniformly on S, then the sequence converges pointwise on S.

2. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Complete the following definitions:
   (a) For any subset $S \subset [a, b]$, define $M(f, S)$ and $m(f, S)$.

   (b) A *partition* P of $[a, b]$ is ...
   (c) The *upper and lower Darboux sums* of f with respect the partition P are ...
   (d) The *upper and lower Darboux integrals* of f are ...
   (e) The function f is *integrable* on $[a, b]$ if ...
   (f) If f is integrable on $[a, b]$, the *value of the integral* of f over $[a, b]$ is ...

3. In class lectures and in the book, the following theorem is discussed:

   Theorem. Every monotonic function f on $[a, b]$ is integrable.

   Give a proof of this theorem in the case when f is increasing. You may use basic results about the Darboux integral which were established in class and in the book.

4. (a) Give the ϵ-δ definition for the continuity of a real-valued function f at a point $x_0 \in \mathbb{R}$.
   (b) Use the ϵ-δ definition of continuity to show that if the real-valued function f is continuous at $x_0 \in \mathbb{R}$ and the real-valued function g is continuous at $f(x_0)$, then the composition $g \circ f$ is continuous at x_0.

5. Recall that if $g : \mathbb{R} \to \mathbb{R}$ is a function and $A \subset \mathbb{R}$, then the *image* of A under g is

$$g(A) = \{g(a) : \ a \in A\}.$$

   Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function.
   (a) Prove that if A is a bounded non-empty set, then $\sup f(A) \le f(\sup A)$.
   (b) Prove that if f is also continuous, then $\sup f(A) = f(\sup A)$.

6. (a) Complete the following definition:
   Let f be a real-valued function defined on a set $S \subset \mathbb{R}$. Then f is uniformly continuous on S if ...

   (b) State the Mean Value theorem.

   (c) Suppose a function $f : \mathbb{R} \to \mathbb{R}$ is differentiable and its derivative satisfies $|f'(x)| < B$ for some $B > 0$ and all $x \in \mathbb{R}$. Prove that f is uniformly continuous on $\mathbb{R}$.

7. Consider the following definition.
   If $f : \mathbb{R} \to \mathbb{R}$ is a function, its *square* $f^2 : \mathbb{R} \to \mathbb{R}$ is defined by $f^2(x) = f(x)^2$.

   In parts (a) and (b), decide if the statements are true or false. If the statement is true, prove it. If the statement is false, give a counterexample.

   (a) If a sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ converges uniformly to the identity function $g(x) = x$, then the sequence of squares (f_n^2) converges uniformly to the function $g^2(x) = x^2$.

   (b) If a sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ converges uniformly to a bounded function $g : \mathbb{R} \to \mathbb{R}$, then the sequence of squares (f_n^2) converges uniformly to g^2.

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8. Recall the following definition:

   > Let f be a real-valued function defined on $S \subset \mathbb{R}$. Then f is *uniformly continuous on* S if for each $\epsilon > 0$ there exists $\delta > 0$ such that $x, y \in S$ and $|x - y| < \delta$ imply $|f(x) - f(y)| < \epsilon$.

   Use this definition to prove the following two statements.

   (a) The function $f(x) = \frac{1}{x}$ is uniformly continuous on the interval $(1, \infty)$.

   (b) The function $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0, \infty)$.

9. (a) Complete the following definition:

   A sequence (s_n) of real numbers is said to *converge* to the real number s provided that ...

   (b) State the triangle inequality (for real numbers).

   (c) Suppose that (s_n) is a sequence of real numbers converging to $s \in \mathbb{R}$, and suppose that (t_n) is a sequence of real numbers converging to $t \in \mathbb{R}$. Prove that $(s_n + t_n)$ converges to $s + t$.

10. (a) Complete the following definition:
    A sequence (s_n) of real numbers is called a *Cauchy sequence* if ...

    (b) Prove that if a sequence (s_n) of real numbers converges to $s \in \mathbb{R}$, then (s_n) is a Cauchy sequence.

11. For this problem, let f be a real-valued function defined on an open interval containing a point $a \in \mathbb{R}$.

    (a) Complete the following definition: We say that f is *differentiable at* a if ...

    (b) Recall that a function f is continuous at a if $\lim_{x \to a} f(x) = f(a)$. Prove that if f is differentiable at a, then f is continuous at a.

    (c) Consider the following statement:

If f is a real valued function which is differentiable at every point in $[0,1]$, then there is an $x_0 \in [0,1]$ so that $f(x) \leq f(x_0)$ for every $x \in [0,1]$.

Is this statement true or false? If it is true, explain why it is true. If it is false, provide a counterexample.

12. (a) State the Mean Value Theorem.

    (b) Let f be a real-valued function which is differentiable on $[a,b]$. Let P be a partition of $[a,b]$. Prove that the Darboux sums of the derivative f' satisfy the inequality

$$L(f',P) \leq f(b) - f(a) \leq U(f',P).$$

13. **Not covered!**

    (a) State the Weierstrass M-test:

    (b) Consider the sequence of functions $g_n(x) = \frac{x^n}{1+x^n}$. Let a be a real number satisfying $0 < a < 1$. Prove that $\sum_{n=1}^{\infty} g_n(x)$ converges uniformly on $[0,a]$.

    (c) Let $g_n(x)$ be as in part (b). Define

$$g(x) = \sum_{n=1}^{\infty} g_n(x) \quad \text{for} \quad x \in [0,1).$$

    Explain why g is continuous on $[0,1)$.
    (You can use the conclusion of part (b) even if you did not get part (b) correct.)

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14. (a) State the Archimedean Property.

    (b) Suppose $0 < a < b$. Prove that there is an $n \in \mathbb{N}$ so that $\frac{1}{n} < a$ and $b < n$. (Your proof should use only the Archimedean property or other basic results such as the Completeness axiom.)

15. (a) State the Bolzano-Weierstrass Theorem.

    (b) Let $f : [0,1] \to \mathbb{R}$ be a continuous function so that there is an $M \in \mathbb{R}$ with

$$M = \sup \ \{f(x) \ : \ x \in [0,1]\}.$$

    Use the Bolzano-Weierstrass Theorem to prove that there is an $x \in [0,1]$ so that $f(x) = M$.

16. Let $f_n : \mathbb{R} \to \mathbb{R}$ the function $f_n(x) = \sin(\frac{x}{n})$. Let $g(x) = 0$ for all $x \in \mathbb{R}$.

    (a) Prove that $f_n \to g$ pointwise on $\mathbb{R}$.

    (b) State the definition of uniform convergence of a sequence of functions $f_n \to g$ on a set $S \subset \mathbb{R}$.

    (c) Prove that $f_n \to g$ uniformly on $[0,M]$ for every $M > 0$.

    (d) Does $f_n \to g$ uniformly on $[0,\infty)$? Prove your answer is correct.

17. Let $f : [0,1] \to \mathbb{R}$ be *increasing*. That is, $f(x) \leq f(y)$ whenever $x < y$. Prove that f is integrable over $[0,1]$. (You may only use basic facts about integrability.)

18. Let $f(x) = \cos(x)$.

(a) State a version of Taylor's theorem.

(b) Compute the Taylor series for f about zero.

(c) Use Taylor's theorem to show that the Taylor series for f converges to f pointwise on $\mathbb{R}$. (You must use Taylor's theorem directly.)

19. Let $f : \mathbb{R} \to \mathbb{R}$ be an integrable function. Suppose that $\lim_{x\to 1^+} f(x) = 7$ and $\lim_{x\to 1^-} f(x) = 3$. Define $F(x) = \int_0^x f(t)\, dt$.

    (a) Suppose $y > 1$. Write $F(y) - F(1)$ as an integral of f.

    (b) Show that there is a constant $c > 0$ so that $F(y)-F(1) \geq 6(y-1)$ whenever $1 < y < 1+c$.

    (c) Similarly, it follows that there is a $c > 0$ so that $F(1) - F(y) < 4(1 - y)$ whenever $1-c < y < 1$. (You do not need to prove this.) Use these two facts to show that F is not differentiable at $x = 1$.

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20. (a) State the Completeness Axiom for $\mathbb{R}$.

    (b) Complete the following definition: A sequence (s_n) of real numbers is said to *converge* to the real number s provided that ...

    (c) Complete the following definition: Suppose $(s_n)_{n\in\mathbb{N}}$ is a sequence. A subsequence of this sequence is ...

    (d) State the Intermediate Value Theorem.

    (e) Complete the following definition: Let f be a real-valued function defined on an open interval containing a point $a \in \mathbb{R}$. We say f is *differentiable* at a, or f has a *derivative* at a, if ...

21. True or False. If the statement is true give a short explanation why. If the statement is false then give a counterexample.

    (a) For any sequence of real numbers (s_n), if (s_n) converges then so does $\big(|s_n|\big)$.

    (b) If (a_n) and (b_n) are non-negative bounded sequences then

    $$\limsup(a_n b_n) = \limsup a_n \cdot \limsup b_n.$$

    (c) If $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ are continuous functions so that $f(a) < g(a)$ and $f(b) > g(b)$ then there is an $x \in (a,b)$ so that $f(x) = g(x)$.

    (d) Any differentiable function $f : (0,1) \to \mathbb{R}$ is uniformly continuous.

    (e) If $f : [0,1] \to \mathbb{R}$ is integrable, then $F(x) = \int_0^x f(t)\, dt$ is differentiable on $(0,1)$.

22. The Archimedean Property states:
    If $a > 0$ and $b > 0$, then for some positive integer n, we have $na > b$.
    Use this result to prove that for any real number $x > 0$ there is an integer n so that $\frac{1}{n} < x < n$.

23. Recall the following:

    - A subset $S \subset \mathbb{R}$ is *closed* if whenever s_n is a convergent sequence of points in S we have $\lim s_n \in S$.
    - The *Bolzano-Weierstrass Theorem* states that every bounded sequence has a convergent subsequence.

We proved that if X is a closed bounded set and $f : X \to \mathbb{R}$ is continuous then f is bounded. Prove this fact (which forms part of the proof that every continuous function on a closed and bounded set attains its maximum).

24. (a) State the Mean Value Theorem.

    (b) Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is *uniformly continuous* on $S \subset \mathbb{R}$ if for each $\epsilon > 0$ there exists $\delta > 0$ such that $x, y \in S$ and $|x - y| < \delta$ imply $|f(x) - f(y)| < \epsilon$. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function and f' is bounded, then f is uniformly continuous on $\mathbb{R}$.

25. Recall that if g is a function, we write $\lim_{x \to x_0^+} g(x) = L$ if for all $\epsilon > 0$ there is a $\delta > 0$ such that $x_0 < x < x_0 + \delta$ implies $|g(x) - L| < \epsilon$.

    Let f be an integrable function on $[a, b]$. For x in $[a, b]$, let $F(x) = \int_a^x f(t)\, dt$. Prove that if f is continuous at $x_0 \in (a, b)$ then

$$\lim_{x \to x_0^+} \frac{F(x) - F(x_0)}{x - x_0} = f(x_0).$$

    (This is part of the proof of the Fundamental Theorem of Calculus.)

26. (a) Suppose $s_n \geq 0$ for all N and $\sum s_n$ converges. Explain why there is an N so that $s_n < 1$ when $n > N$.

    (b) State the Comparison test for series.

    (c) Suppose s_n is a non-negative sequence of real numbers and suppose $\sum s_n$ converges. Prove that $\sum s_n^2$ converges.
    Hints: You can use (a) and (b) to prove this. It suffices to prove that $\sum_{n=N+1}^{\infty} s_n$ converges for some $N \in \mathbb{N}$.