Math 323: Practice for Midterm 3

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Disclaimer. This test is just a recommendation of things to study and problems to work on. You may be asked about things that do not appear here. You should practice doing problems from the book in addition to the problems included in this sheet.

Covered Material. Material explicitly covered will include §19, 23, 24, 28, 29. Also, you are expected to know all material covered in the course up until now.

Definitions. You will be asked to define several terms on the test. **These terms all have one definition, as given in the book. You are expected to know this definition.** The following is a list of terms which might appear. (Others might appear as well).

uniformly continuous, closed set, extension of a function, $\lim_{x \to a^S} f(x)$*, limit of a function at a point, right-hand (and left-hand) limit of a function at a point, power series, radius of convergence, interval of convergence, converges pointwise, converges uniformly, differentiable at a, differentiable, derivative, strictly increasing, increasing, strictly decreasing, decreasing,*

Theorems. Theorems given names in the book are often the most important. Theorems (and similar results) you may be required to state:

28.4 Chain Rule, 29.2 Rolles Theorem, 29.3 Mean Value Theorem, 29.8 Intermediate Value Theorem for Derivatives

Problems. I am presenting the following problems because they would be good practice. In particular, they do not necessarily represent problems that I would give on a test.

1. (a) (8 points) Complete the following definition:

   A real valued function $f : \mathbb{R} \to \mathbb{R}$ is *strictly increasing* if...

   (b) (8 points) State the Mean Value Theorem.

   (c) (8 points) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $f'(x) > 0$ for all $x \in \mathbb{R}$, then f is strictly increasing.

2. Consider the following definitions:

   Let f be a real-valued function defined on an open interval containing $a \in \mathbb{R}$. The function is *rightward increasing at* a if there is a $\delta > 0$ so that $a < x < a + \delta$ implies that $f(x) \geq f(a)$. Similarly, the function is *rightward decreasing at* a if there is a $\delta > 0$ so that $a < x < a + \delta$ implies that $f(x) \leq f(a)$.

   (a) Prove that $f(x) = x - x^3$ is rightward increasing at $a = 0$.

   (b) Prove that the following function is neither rightward increasing nor rightward decreasing at $a = 0$:

   $$f(x) = \begin{cases} x \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

3. Let $f : \mathbb{R} \to \mathbb{R}$ and suppose f is differentiable at zero with $f'(0) > 0$. Show that there is an $\epsilon > 0$ so that whenever $0 < x < \epsilon$, we have $f(x) > f(0)$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ and suppose f is differentiable at zero with $f'(0) > 0$. Show that there is an $\epsilon > 0$ so that whenever $0 < x < \epsilon$, we have $f(x) > f(0)$.

5. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function. Also assume that

$$\lim_{x \to +\infty} f(x) = \lim_{x \to -\infty} f(x) = +\infty.$$

   (a) Prove that there is an x so that $f'(x) = 0$.

   (b) Prove that there is an x so that $f'(x) > 0$.

   (c) Prove that there is an x so that $f'(x) < 0$.

   (d) Given an example of a function f satisfying the statements above for which $-1 < f'(x) < 1$ for all $x \in \mathbb{R}$.

6. Let f be a differentiable function defined on $[\frac{1}{4}, \frac{3}{4}]$ so that $f(\frac{1}{4}) = f(\frac{3}{4})$. For each $a \in \mathbb{R}$, define $g_a(x) = ax(1-x)$. We will prove that the graph of f is somewhere tangent to the graph of g_a for some a. (That is, there is an a and an $x \in (\frac{1}{4}, \frac{3}{4})$ so that $f(x) = g_a(x)$ and $f'(x) = g_a'(x)$.)

   (a) Define $h(x) = \frac{f(x)}{x(1-x)}$. Observe that $f(x) = g_a(x)$ whenever $a = h(x)$.

   (b) Show that there is an $x_0 \in (\frac{1}{4}, \frac{3}{4})$ with $h'(x_0) = 0$.

   (c) With x_0 as in part (b), show that there is an a so that $f(x_0) = g_a(x_0)$ and $f'(x_0) = g_a'(x_0)$.

7. (14 points) Let $f(x) = x\sqrt{|x|}$. (This is the product of x and the square root of the absolute value of x.) At which points $x \in \mathbb{R}$ is f differentiable? Prove your answer is correct, and rigorously compute the derivative at all points at which f is differentiable.

8. (12 points) Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is *strictly increasing* if for any $x_1, x_2 \in \mathbb{R}$, $x_1 < x_2$ implies $f(x_1) < f(x_2)$. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $f'(x) > 0$ for all x, then f is strictly increasing.

9. State if the following statements are true or false. If true, briefly explain why. If false, give a counterexample.

   (a) (6 points) If the function $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous on each interval of the form $[n, n+1]$ with $n \in \mathbb{Z}$, then it is uniformly continuous on $\mathbb{R}$.

   (b) (6 points) A uniformly continuous function $f : \mathbb{R} \to \mathbb{R}$ attains its maximum. (The book says "assumes its maximum" for the same notion.)

10. (a) (8 points) Complete the following definition:
    Let f be a real-valued function defined on a set $S \subset \mathbb{R}$. Then f is *uniformly continuous* on S if ...

    (b) (14 points) Let $a > 0$ be an arbitrary positive real number. Prove that $f(x) = \frac{1}{x}$ is uniformly continuous on (a, ∞).

11. (a) (8 points) Complete the following definition:
    Let (f_n) be a sequence of real-valued functions defined on a set $S \subset \mathbb{R}$. The sequence (f_n) *converges uniformly* on S to a function f defined on S if ...

    (b) (14 points) Suppose that $f_n : \mathbb{R} \to \mathbb{R}$ is a sequence of functions which converges uniformly on $\mathbb{R}$ to $f : \mathbb{R} \to \mathbb{R}$. Suppose that $g : \mathbb{R} \to \mathbb{R}$ is uniformly continuous on $\mathbb{R}$. Prove that the sequence of compositions $(g \circ f_n)$ converges uniformly on $\mathbb{R}$ to the function $g \circ f$.

12. (a) (8 points) Complete the following definition:
    Let f be a real-valued function defined on a set $S \subset \mathbb{R}$. Then f is uniformly continuous on S if ...
    (b) (8 points) State the Mean Value theorem.
    (c) (12 points) Suppose a function $f : \mathbb{R} \to \mathbb{R}$ is differentiable and its derivative satisfies $|f'(x)| < B$ for some $B > 0$ and all $x \in \mathbb{R}$. Prove that f is uniformly continuous on $\mathbb{R}$.

13. Consider the following definition.
    If $f : \mathbb{R} \to \mathbb{R}$ is a function, its *square* $f^2 : \mathbb{R} \to \mathbb{R}$ is defined by $f^2(x) = f(x)^2$.

    In parts (a) and (b), decide if the statements are true or false. If the statement is true, prove it. If the statement is false, give a counterexample.
    (a) (10 points) If a sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ converges uniformly to the identity function $g(x) = x$, then the sequence of squares (f_n^2) converges uniformly to the function $g^2(x) = x^2$.
    (b) (10 points) If a sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ converges uniformly to a bounded function $g : \mathbb{R} \to \mathbb{R}$, then the sequence of squares (f_n^2) converges uniformly to g^2.

14. (a) (10 points) Complete the following definition.
    Let f be a real-valued function defined on a set $S \subset \mathbb{R}$. Then f is **uniformly continuous on** S *if ...*
    (b) (15 points) Use this definition to prove that the function $f(x) = x^3$ is not uniformly continuous on $\mathbb{R}$.

15. Suppose that (a_n) is a sequence satisfying $\frac{1}{n+2} < a_n < \frac{1}{n+1}$ for all integers $n \geq 0$. Consider the power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = \lim_{N\to\infty} a_0 + a_1 x + \ldots + a_N x^N.$$

    (a) Show that the radius of convergence of the power series is one.
    (b) Prove that the series converges when $x = -1$.
    (c) Prove that the series diverges when $x = 1$.