Math 323: Practice for Midterm 2

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Disclaimer. This test is just a recommendation of things to study and problems to work on. You may be asked about things that do not appear here. You should practice doing problems from the book in addition to the problems included in this sheet.

Covered Material. Material explicitly covered will include §11-15, §17-18, and §20. Section 13 (on open and closed sets) will only be covered in the context of $\mathbb{R}$. Earlier sections may also be covered especially material in §7-10, which is basic material about sequences. You are expected to know all material covered in the course up until now.

Definitions. You will be asked to define several terms on the test. **These terms all have one definition, as given in the book. You are expected to know this definition.** The following is a list of terms which might appear. (Others might appear as well).

subsequence, subsequential limit, limit superior ($\limsup$*), and limit inferior (*$\liminf$*), Cauchy criterion, absolutely convergent, absolutely convergent, alternating series, continuous, continuous at a point, continuous on a set, discontinuous, composition, limit of a function at a point, right-hand limit of a function at a point*

Theorems. Theorems given names in the book are often the most important. Theorems (and similar results) you may be required to state:

11.5: Bolzano-Weierstrass Theorem, 14.6 Comparison Test, 14.8: Ratio Test, 14.9: Root Test, 15.2: Integral tests, 15.3: Alternating Series Theorem, 17.2: δ-ϵ definition of continuity, 18.1: Continuous functions on closed and bounded sets attain their maximum and minimum, 18.2: Intermediate Value Theorem

Problems. I am presenting the following problems because they would be good practice. In particular, they do not necessarily represent problems that I would give on a test.

1. Suppose that (a_n) and (b_n) are sequences satisfying $a_n \geq 0$ and $b_n \geq 0$ for all $n \in \mathbb{N}$. Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge. Prove that $\sum_{n=1}^{\infty} a_n b_n$ converges.

2. Suppose that (a_n) is a sequence which satisfies the following two statements.

   1. $|a_n + a_{n+1}| < \frac{1}{2^n}$ for all $n \in \mathbb{N}$.
   2. $\lim_{n\to\infty} a_n = 0$.

   (a) Prove that $\sum_{n=1}^{\infty} a_n$ converges.

   (b) Give an example of a sequence (a_n) so that $\sum_{n=1}^{\infty} a_n$ does not converge, but (a_n) still satisfies statement 1 above.

3. Recall that a *counterexample* is an example which proves that a statement is false. Give counterexamples to the following false statements.

   (a) Every convergent series is absolutely convergent.

   (b) If (a_n) is a monotone sequence which converges to zero, then the series $\sum_{n=1}^{\infty} a_n$ converges.

4. (a) State the **Cauchy criterion** for convergence of the series $\sum_{n=1}^{\infty} a_n$.

(b) Suppose that the series $\sum_{n=1}^{\infty} a_n$ converges. Prove that $\lim_{n\to\infty} a_n = 0$.

5. Define the sequence $s_n = \sin(n)$.
   (a) Prove that the sequence (s_n) has a convergent subsequence.
   (b) Prove that this sequence has no subsequence (s_{n_k}) which converges to 2.

6. Give an example of a sequence satisfying the property or explain why such a sequence does not exist:
   A sequence whose set of subsequential limits is exactly $S = \{\frac{1}{n} \ : \ n \in \mathbb{N}\}$.

7. Determine if the series $\sum_{n=1}^{\infty} \frac{n^4}{2^n}$ converges. Justify your answer. (You may use any theorem you like.)

8. State whether each of the following statements is true or false. If the statement is true, briefly explain why. If the statement is false, give a counterexample. Your explanations and counterexamples need not be more than a sentence or two.
   (a) If (a_n) is any sequence of real numbers and the series $\sum |a_n|$ converges, then the series $\sum a_n$ converges.
   (b) If (a_n) is any sequence so that $\lim a_n = 0$, then $\sum a_n$ converges.
   (c) If (a_n) is any sequence of positive numbers and $\lim \frac{a_{n+1}}{a_n} = 1$, then $\sum a_n$ converges.
   (d) If (a_n) is any sequence of positive numbers and $\sum a_n$ converges, then $\sum a_n^2$ converges.

9. (a) State the Bolzano-Weierstrass Theorem.
   (b) Let (s_n) be a sequence of real numbers so that there are infinitely many $n \in \mathbb{N}$ with $|s_n| < 1$. Prove that (s_n) has a convergent subsequence.

10. (a) Suppose $S \subset \mathbb{R}$ and $\sup S = M$. Prove that if $N < M$ then there is an $s \in S$ so that $s > N$.
    (b) Suppose (s_n) is a sequence and $\limsup s_n = x \in \mathbb{R}$. Prove that if $y < x$ then there is an n so that $s_n > y$.

11. (a) Suppose (a_n) and (b_n) are sequences with $\limsup a_n = a \in \mathbb{R}$ and $\limsup b_n = b \in \mathbb{R}$. Prove that $\limsup(a_n + b_n) \leq a + b$.
    (b) Give an example of sequences (a_n) and (b_n) where

$$\limsup(a_n + b_n) < \limsup a_n + \limsup b_n.$$

12. (a) Complete the ϵ-δ definition of continuity.
    Let f be a real-valued function whose domain is a subset of $\mathbb{R}$. Then f is **continuous** *at $x_0 \in dom(f)$ if and only if ...*
    (b) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ x^2 & \text{if } x < 0. \end{cases}$$

    Use the ϵ-δ definition to prove that f is continuous at zero.

13. (a) State the ϵ-δ definition of continuity of a function f at a point x_0.
    (b) Use the ϵ-δ definition to directly prove that $f(x) = x + x^3$ is continuous at 0. (You may not use other results you know. Just verify the definition.)

14. (a) State the ϵ-δ definition of **continuity** of a function $f : \mathbb{R} \to \mathbb{R}$ at a point $x_0 \in R$.
    (b) Let f and g be real-valued continuous functions defined on $\mathbb{R}$. Let x_0 be a real number and suppose that $f(x_0) < g(x_0)$. Prove that there is an open interval (a, b) containing x_0 so that $f(x) < g(x)$ for every $x \in (a, b)$.

15. (a) State the Intermediate Value Theorem.
    (b) Suppose $f : [0, 1] \to \mathbb{R}$ and $g : [0, 1] \to \mathbb{R}$ are continuous functions so that $f(x) \neq g(x)$ for all $x \in [0, 1]$. Prove that if $f(x_0) < g(x_0)$ for some $x_0 \in [0, 1]$, then $f(x) < g(x)$ for every $x \in [0, 1]$.

16. (a) (10 points) State the Intermediate Value Theorem.
    (b) (15 points) Suppose that f is a polynomial of degree four of the form

$$f(x) = x^4 + ax^3 + bx^2 + cx + d,$$

    with $a, b, c, d \in \mathbb{R}$. Also suppose there is a $y \in \mathbb{R}$ so that $f(y) < 0$. Prove that f has a real root. That is, prove that there is an $x_0 \in \mathbb{R}$ so that $f(x_0) = 0$.

17. Consider the following definitions:

    Let f be a real-valued function defined on an open interval containing $a \in \mathbb{R}$. The function is *rightward increasing at a* if there is a $\delta > 0$ so that $a < x < a + \delta$ implies that $f(x) \geq f(a)$. Similarly, the function is *rightward decreasing at a* if there is a $\delta > 0$ so that $a < x < a + \delta$ implies that $f(x) \leq f(a)$.

    (a) Prove that $f(x) = x - x^3$ is rightward increasing at $a = 0$.
    (b) Prove that the following function is neither rightward increasing nor rightward decreasing at $a = 0$:

$$f(x) = \begin{cases} x \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

18. There are two definitions of continuity at a point. (The book calls one the ϵ-δ definition of continuity but states it as a Theorem.) Complete the definition in these two ways.
    (a) (8 points) The function f is continuous at x_0 in dom(f) if, ...
    (b) (8 points) The function f is continuous at x_0 in dom(f) if, ...
    (c) (12 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a function which is continuous and non-negative (i.e., $f(x) \geq 0$ for every $x \in \mathbb{R}$). Suppose $g : \mathbb{R} \to \mathbb{R}$ is another function which satisfies $|g(x)| \leq f(x)$ for every $x \in \mathbb{R}$. Prove that if $f(x_0) = 0$ for some $x_0 \in \mathbb{R}$, then g is continuous at x_0.

19. (a) State the ϵ-δ definition of **continuity** of a function $f : \mathbb{R} \to \mathbb{R}$ at a point $x_0 \in R$.
    (b) Let f and g be real-valued continuous functions defined on $\mathbb{R}$. Let x_0 be a real number and suppose that $f(x_0) < g(x_0)$. Prove that there is an open interval (a, b) containing x_0 so that $f(x) < g(x)$ for every $x \in (a, b)$.

20. Recall that if $g : \mathbb{R} \to \mathbb{R}$ is a function and $A \subset \mathbb{R}$, then the *image* of A under g is

$$g(A) = \{g(a) : \ a \in A\}.$$

Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function.

(a) (10 points) Prove that if A is a bounded non-empty set, then $\sup f(A) \leq f(\sup A)$.

(b) (10 points) Prove that if f is also continuous, then $\sup f(A) = f(\sup A)$.