Math 323: Practice for Midterm 1

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Disclaimer. This test is just a recommendation of things to study and problems to work on. You may be asked about things that do not appear here. You should practice doing problems from the book in addition to the problems included in this sheet.

Covered material. The midterm will cover §1-5 and §7-10. You will not be asked to prove certain numbers are irrational (which is most of the content of §2).

Definitions. You will be asked to define several terms on the test. **These terms all have one definition, as given in the book. You are expected to know this definition.** The following is a list of terms which might appear. (Others might appear as well).

absolute value, maximum, minimum, upper bound, lower bound, supremum, infimum, converge, diverge, limit, bounded sequence, nonincreasing sequence, nondecreasing sequence, monotone sequence, Cauchy sequence

Theorems. Theorems given names in the book are the most important. Theorems (and similar results) you may be required to state:

4.4: Completeness Axiom, 4.6: Archimedian Property.

Problems. These problems are intended to prepare you for the midterm, though not all of them are problems I would put on a midterm.

1. Suppose (s_n) is a sequence of real numbers such that $s_n < 0$ when n is odd, and $s_n > 1$ when n is even. Prove that the sequence (s_n) does not converge to a real number.

2. (a) State the Archimedian property.
   (b) (15 points) Use the Archimedian property to prove that if a and b are real numbers with $0 < a < b$, then there is an $n \in \mathbb{N}$ such that $\frac{1}{n} < a$ and $b < n$.

3. Define the sequence (s_n) inductively by setting $s_1 = 2$ and $s_{n+1} = \frac{n(n+2)}{(n+1)^2} s_n$ for all $n \in \mathbb{N}$.
   (a) Find a formula for s_n in terms of n. Use induction to prove your answer is correct.
   (b) Does s_n converge? If so, what do you think the limit is?
   (c) Use the definitions of *convergence* and *limit* to prove that your answer to part (b) is correct.

4. Give examples of the following.
   (a) A bounded sequence which does not converge.
   (b) A non-empty set S so that $\inf S \in S$ but $\sup S \notin S$.

5. (a) State the Archimedian property.

(b) Use the Archimedean property to prove that if $a, b \in \mathbb{R}$ with $0 \le a < b$, then there is a rational number $\frac{p}{q}$ with p and q both odd such that $a < p/q < b$. (*Remark:* Modify the proof that rationals are dense. This would be too long for a midterm!)

6. Let (s_n) be a sequence of real numbers, and suppose that

$$\lim_{n\to\infty} s_{n+1} - s_n = 1.$$

Prove that $\lim_{n\to\infty} s_n = +\infty$. (*Hints*: Use the observations that eventually $s_{n+1} - s_n > \frac{1}{2}$ and that for each pair of integers n and m satisfying $n \ge m > 0$, we have $s_n = s_m + \sum_{j=m}^{n-1} s_{j+1} - s_j$.)

7. (a) Complete the following definition. "Let S be a nonempty subset of $\mathbb{R}$. We call s_0 the *maximum* of S if..."

(b) Complete the following definition. "A nonempty subset S of $\mathbb{R}$ is bounded if..."

(c) Suppose (s_n) is a strictly increasing sequence of real numbers. (*Strictly increasing* means that $s_{n+1} > s_n$ for every $n \in \mathbb{N}$.) Prove that the set S consisting of all elements in the sequence has no maximum.

8. (a) Complete the following definition: "A sequence (s_n) of real numbers converges to $L \in \mathbb{R}$ if..."

(b) Prove the following theorem:
Squeeze Theorem. Suppose (a_n), (b_n) and (c_n) are three sequences of real numbers so that $a_n \le b_n \le c_n$ for all $n \in \mathbb{N}$ and there is a real number L so that

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n.$$

Then, $\lim_{n\to\infty} b_n = L$.

9. Prove the following result by following the steps listed below.
Proposition: If α and β are real numbers with $0 < \alpha \le \beta$, then there is an integer n so that

$$|n\alpha - \beta| \le \frac{\alpha}{2}.$$

(a) How do you know that there is an integer $m > 0$ so that $m\alpha > \beta$?

(b) Use part (a) to prove that there is a smallest integer so that $n\alpha > \beta$.

(c) (8 points) Let n be the integer found in part (b). Prove that either

$$0 \le n\alpha - \beta \le \frac{\alpha}{2} \quad \text{or} \quad \frac{-\alpha}{2} \le (n-1)\alpha - \beta \le 0.$$

This will prove the proposition above, because in the first case $|n\alpha - \beta| \le \frac{\alpha}{2}$ and in the second case $|(n-1)\alpha - \beta| \le \frac{\alpha}{2}$.

10. Define a sequence (s_n) inductively by the rules $s_1 = 1$ and $s_{n+1} = s_n + \frac{1}{2^n}$ for integers $n \ge 1$. Use induction to prove that $s_n = 2 - \frac{1}{2^{n-1}}$ for each integer $n \ge 1$.

11. (a) Complete the following definition:
    A sequence (s_n) of real numbers is said to converge to the real number s provided that...

    (b) Use the definition to prove the following result:
    Suppose (s_n) and (t_n) are sequences of real numbers. If (s_n) converges to s and (t_n) converges to t, then $(s_n + t_n)$ converges to $s + t$.

12. Suppose $0 < p < 1$. Consider the sequence defined inductively by $s_1 = 1$, and $s_{n+1} = ps_n + 1$ for $n \in \mathbb{N}$.

    (a) Use induction to prove that $s_n > 0$ for all $n \in \mathbb{N}$.

    (b) Prove that $\frac{1}{1-p}$ is an upper bound for (s_n).

    (c) Use this upper bound to prove that (s_n) is increasing.

    (d) Can you conclude that (s_n) converges? Why or why not? If it does converge, what is the limit? Why?

13. Suppose (s_n) is a sequence of real numbers such that $s_n < 0$ when n is odd, and $s_n > 1$ when n is even. Prove that the sequence (s_n) does not converge to a real number by proving that (s_n) is not Cauchy.

14. Give examples of the following.

    (a) A bounded sequence which does not converge.

    (b) A monotone sequence which diverges.

15. Let (a_n) be a sequence of real numbers which is both non-decreasing and bounded. Give the proof that the sequence converges.