Math 308: Practice for Final Exam

Prof. Hooper

Disclaimer. This test is just a recommendation of things to study and problems to work on. You may be asked about things that do not appear here. You should practice doing problems from the book in addition to the problems included in this sheet.

Covered material. From the Gerstein book: §1.1-1.5, §2.1-2.10, §3.1-3.3 and §4.1-4.2. From Ross: §1-4. The final exam will cover everything taught in the course.

Definitions. You will be asked to define several terms on the test. **These terms all have one definition, as given in the book. You are expected to know this definition.** The following is a list of terms which might appear. (Others might appear as well).

From Gerstein: *axioms, proposition, truth value, prime number, truth functional, connective, truth table, conjunction, statement forms (or compound statement), statement letters, disjunction, inclusive or, exclusive or, conditional, implication, hypothesis, conclusion, converse, tautology, modus ponens, direct proof, proof by contradiction, contrapositive, logically equivalent, logically equivalence, De Morgans laws, contradiction, set, member, element, equality of sets, natural numbers $\mathbb{N}$, integers $\mathbb{Z}$, rational numbers $\mathbb{Q}$, real numbers $\mathbb{R}$, irrational numbers (i.e., real numbers that are not rational), even integer, odd integer, empty set, variable, existential quantifier, universal quantifier, universal set, subset, proper subset, set difference, complement, universal set, union, intersection, disjoint, associative laws (for sets), distributive laws (for sets), De Morgans laws (for sets), indexes, index set, indexed set, union/intersection of an indexed family of sets, disjoint (for indexed family of sets), pairwise disjoint, power set, ordered pair (know the "quasi-definition"), first and second coordinates, Cartesian product, partition, blocks of a partition, relation, reflexive, symmetric, transitive, equivalence relation, equivalent, equivalence class, partition induced by an equivalence relation, equivalence relation induced by a partition, finer, well-ordering principle, basis step, induction step, induction hypothesis, multiple, function (also known as a mapping, map or transformation), domain, image, pre-image, codomain, identity map, constant function, graph, equal functions, restriction, range (or image), surjective (or onto), injective (or one-to-one), bijection, finite sequence, infinite sequence, n-tuple, composition, inverse of a function, left inverse, right inverse, equipotent (or same cardinality), finite set, infinite set, count, (really a theorem), Cartesian product, cardinality less than*

From Ross: *associative laws, commutative laws, distributive law, field, absolute value, maximum, minimum, upper bound, lower bound, supremum, infimum*

Theorems, Axioms, etc. Results given names in the books are important. Theorems (and similar results) you should know:

From Gerstein: *Axiom of Separation (page 41), 3.23: Associative Law of Function Composition, 3.28: Cancellation Law, 4.9: The Pigeonhole Principle, 4.28: Schröder-Bernstein Theorem, 4.31: Cantors Theorem (I), 4.32: Cantors Theorem (II), The Fundamental Theorem of Arithmetic (page 169), 4.41: Cantors Theorem (II),*
From Ross: *4.4: Completeness Axiom, 4.6: Archimedian Property.*

Problems. These problems are intended to prepare you for the final. Also consider the exercises in the books (both assigned and unassigned).

1. (a) State the Completeness axiom.

   (b) Suppose that $S \subset \mathbb{R}$ is a set satisfying the following statements:

      (i) The set S contains a positive real number.

      (ii) If $s \in S$, then there is a $t \in S$ with $t > 2s$.

   Prove that S does not have an upper bound. (*Hint*: Try proof by contradiction.)

2. Give a proof by contradiction that $\sqrt[3]{10}$ is irrational. (Do not use the Rational Zeros Theorem).

3. Complete the following definitions:

   (a) A relation R on a set A is *reflexive* if ...

   (b) A relation R on a set A is *symmetric* if ...

   (c) Two sets A and B have the *same cardinality* if ...

   (d) A set A is *countably infinite* if

   (e) A function $f : X \to Y$ is *onto* or *surjective* if ...

4. Provide short answers to the following questions. Proofs are not needed here.

   (a) Give an explicit bijection from $\mathbb{N} \to \mathbb{Z}$.

   (b) Explain what the fact that there is a bijection from $\mathbb{N} \to \mathbb{Z}$ says about the cardinality of $\mathbb{Z}$.

5. (a) State the Archimedean Property.

   (b) Let $a > 0$ be a real number. Give a formal proof that there is a natural number n so that $\frac{1}{n} < a < n$.

6. Change quantifiers and logical operators to answer the following (as taught in class).

   (a) Let $n \in \mathbb{N}$. Negate the statement:
   If n is odd, then n^2 is even.

   (b) Let m be an integer. Negate the statement:
   We have $m^2 \equiv 1 \pmod 3$ or $m \equiv 0 \pmod 3$.

   (c) Let $f : A \to B$ be a function. Negate the statement:
   For every $b \in B$, there is an $a \in A$ so that $f(a) = b$.

   (d) Let $S \subset \mathbb{R}$. Negate the statement:
   For all $s \in S$, there are integers m and n so that $s = \frac{m}{n}$.

   (e) Let R be a relation on a set X. Negate the statement:
   For every $x, y, z \in X$, if xRy and yRz then xRz.

   (f) Let $S \subset \mathbb{N}$. State the contrapositive of the following implication:
   If $n \in S$, then $n + 1 \in S$.

(g) Let A and B be sets. State the contrapositive of the following implication:
If $B \subset A$ and B is infinite, then A and B have the same cardinality.

(h) Let x be a real number. State the converse of the following implication:
If $x > 0$, then $x^2 \geq x$.

7. (a) Let a and b be integers. Prove that if ab is even, then a is even or b is even.

   (b) Use part (a) to prove that if c is an odd integer, then $\sqrt{2c}$ is irrational.

8. Prove that for every natural number n,

$$\sum_{k=1}^{n} \frac{1}{2^k} = 1 - \frac{1}{2^n}.$$

9. Recall the following definition:
A relation R on a set A is *transitive* if for every $a, b, c \in A$, aRb and bRc implies aRc.

Each of the following three parts defines a relation on a set. In each case, either prove that the relation is transitive, or prove that it is not transitive by providing a counterexample.

(a) The relation S on $\mathbb{R}$ is defined by xSy if there is an $n \in \mathbb{Z}$ so that $nx = y$.

(b) The relation R on $\mathbb{Z}$ is defined by aRb if there is an integer $n \geq 2$ so that $n|a$ and $n|b$.

(c) The relation $\sim$ on the set $\mathcal{P}(\mathbb{R})$ is defined by $A \sim B$ if $A \cap B \neq \emptyset$. (**Remark:** $\mathcal{P}(\mathbb{R})$ denotes the power set of $\mathbb{R}$, i.e., the set of all subsets of $\mathbb{R}$.)

(d) Let $n \geq 2$ be an integer. The relation T on the set $\mathbb{Z}$ is defined by aTb if n divides $a - b$.

10. (a) Complete the following definition:
A function $f : X \to Y$ is *one-to-one* or *injective* if ...

    (b) Complete the following definition:
If $f : A \to B$ and $g : B \to C$ are functions, their *composition* is ...

    (c) Prove that if $f : A \to B$ and $g : B \to C$ are injective functions, then their composition is also injective.

11. Recall that a *field* is a set F together with special elements $0, 1 \in F$ and binary operations $+$ and $\cdot$ which satisfy the statements:

A1. $a + (b + c) = (a + b) + c$ for all $a, b, c \in F$.

A2. $a + b = b + a$ for all $a, b \in F$.

A3. $a + 0 = a$ for all $a \in F$.

A4. For each $a \in F$, there is an element $-a \in F$ such that $a + (a) = 0$.

M1. $a(bc) = (ab)c$ for all $a, b, c \in F$.

M2. $ab = ba$ for all $a, b \in F$.

M3. $a \cdot 1 = a$ for all $a \in F$.

M4. For each $a \in F$ with $a \neq 0$, there is an element a^{-1} such that $aa^{-1} = 1$.

DL. $a(b + c) = ab + ac$ for all $a, b, c \in F$.

Use the definition of a field to prove the following two statements. Be sure to describe when you are using each of the statements above.

(a) For each $a, b, c \in F$, $a + c = b + c$ implies $a = b$.

(b) For each $a \in F$, $a \cdot 0 = 0$.

12. Give counterexamples to the following false statements.
    (a) Every relation which is reflexive and symmetric is transitive.
    (b) Every infinite set is denumerable.
    (c) For every real number $x \geq 0$, $x^2 \geq x$.
    (d) Every two countable sets have the same cardinality.
    (e) Let $m, n \in \mathbb{Z}$. If $mn \equiv 0 \pmod 6$, then $m \equiv 0 \pmod 6$ or $n \equiv 0 \pmod 6$.
    (f) Every non-empty bounded subset of $\mathbb{R}$ has a maximum.
    (g) Every injective function is surjective.
    (h) Each bounded set of real numbers with a minimum also has a maximum.
    (i) Whenever the sets $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$ are bounded from above, the set consisting of all products of elements of A with elements of B,
    $$A \cdot B = \{ab : \ a \in A \text{ and } b \in B\},$$
    is also bounded from above.

13. Let A, B and C be sets. Prove that $(A \setminus B) \cup (A \setminus C) = A \setminus (B \cap C)$.

14. Let A and B be sets. Prove that $A \cup B = A \cap B$ if and only if $A = B$.

15. Prove that for all positive integers n, $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$.

16. Prove that $2^n > n^2$ for every integer $n \geq 5$.

17. Let R and S be equivalence relations on a set X. Prove that the relation T on X defined by xTy if xRy and xSy is an equivalence relation. (That is, prove that the intersection of two equivalence relations is an equivalence relation.)

18. Define a relation $\sim$ on $\mathbb{R}$ by $x \sim y$ if $|x - y| \leq 1$. Is $\sim$ an equivalence relation? Prove your assertion is correct.

19. Let $f : A \to B$ and $g : B \to C$ be functions. Prove that if $g \circ f$ is surjective, then g is surjective.

20. Let S and T be sets. Prove that if $T \setminus S$ and $S \setminus T$ have the same cardinality, then so do S and T.

21. Complete the following definitions.

    (a) Let S be a non-empty subset of $\mathbb{R}$. A real number $s_0 \in \mathbb{R}$ is the *maximum* of S if ...

    (b) Let S be a non-empty subset of $\mathbb{R}$. A real number $s_0 \in \mathbb{R}$ is the *supremum* of S if ...

22. Give counterexamples to the following false statements. You do not need to prove your answer is correct.

    (a) Every two countable sets have the same cardinality.

    (b) Let $m, n \in \mathbb{Z}$. If mn is a multiple of six, then m is a multiple of six or n is a multiple of six.

    (c) Every non-empty bounded subset of $\mathbb{R}$ has a maximum.

23. Let $a, b \in \mathbb{R}$. Prove that if $a \leq c$ for every $c > b$, then $a \leq b$.

24. (a) Complete the following definition:
    If $S \subset \mathbb{R}$ is non-empty and $M \in \mathbb{R}$, then M is an upper bound for S if ...

    (b) Suppose S and T are bounded subset of $\mathbb{R}$ satisfying $S \subset T$. Prove that $\sup S \leq \sup T$.