Math 308: Practice for Midterm 3

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Disclaimer. This test is just a recommendation of things to study and problems to work on. You may be asked about things that do not appear here. You should practice doing problems from the book in addition to the problems included in this sheet.

Covered material. The midterm will cover §3.1-3.3 and §4.1-4.2. You are also expected to know prior material.

Definitions. You will be asked to define several terms on the test. **These terms all have one definition, as given in the book. You are expected to know this definition.** The following is a list of terms which might appear. (Others might appear as well).

function (also known as a mapping, map or transformation), domain, image, pre-image, codomain, identity map, constant function, graph, equal functions, restriction, range (or image), surjective (or onto), injective (or one-to-one), bijection, finite sequence, infinite sequence, n-tuple, composition, inverse of a function, left inverse, right inverse, equipotent (or same cardinality), finite set, infinite set, count, The Pigeonhole Principle (really a theorem), Cartesian product, cardinality less than

Problems. These problems are intended to prepare you for the midterm, though not all of them are problems I would put on a midterm. Also consider the exercises in the book (both assigned and unassigned).

1. (a) Complete the following definition:
   A function $f : A \to B$ is *onto* or *surjective* if ...

   (b) Use the definition to prove that if $f : A \to B$ and $g : B \to C$ are both surjective, then so is their composition $g \circ f : A \to C$.

2. Prove that the intervals $(0, 1)$ and $(1, \infty)$ have the same cardinality.

3. (a) Complete the following definition:
   A function $f : A \to B$ is *one-to-one* or *injective* if ...

   (b) Suppose $f : A \to B$ and $g : B \to C$ are injective maps. Use the definition to prove that their composition $g \circ f : A \to C$ is injective. (You will loose points for other methods.)

4. (a) Prove that if a is a natural number then $\mathbb{N} \approx \mathbb{N} \setminus \{a\}$.

   (b) Explain that if $S \approx \mathbb{N}$ and $s \in S$, then $\mathbb{N} \approx S \setminus \{s\}$.

5. Provide short answers to the following questions. Proofs are not needed here.

   (a) Give an explicit bijection from $\mathbb{N} \to \mathbb{Z}$.

   (b) Explain what the fact that there is a bijection from $\mathbb{N} \to \mathbb{Z}$ says about the cardinality of $\mathbb{Z}$.

6. Give counterexamples to the following false statements.

(a) Every infinite set has the same cardinality as $\mathbb{N}$.

(b) Every two finite sets have the same cardinality.

(c) Every injective function is surjective.

7. Let $f : A \to B$ and $g : B \to C$ be functions. Prove that if $g \circ f$ is injective, then f is injective.

8. Let $f : A \to B$ be a function. Recall that a function $g : B \to A$ is the *inverse* of $f : A \to B$ if $g \circ f(a) = a$ for all $a \in A$ and $f \circ g(b) = b$ for all $b \in B$. Prove that if f has an inverse, g, then f is one-to-one and onto.

9. Let S and T be sets. Prove that if $T \setminus S$ and $S \setminus T$ have the same cardinality, then so do S and T.

10. Let X be the collection of all finite subsets of $\mathbb{N}$. Prove that $X \approx \mathbb{N}$. (*Hint:* Construct an explicit bijection from X to $N_0 = \{0, 1, 2, 3, \ldots\}$. You can use the fact that every $n \in N_0$ can be expressed in a unique way as $n = \sum_{k=0}^{\infty} \delta_k 2^k$ where each $\delta_k \in \{0, 1\}$.) (*Remark*: This is a harder question than something which would appear on the midterm.)

11. Complete the proof of the following result.
    Result: Let S be a nonempty set and $\mathcal{P}(S)$ be its power set. Then $\#S \neq \#\mathcal{P}(S)$.

    Proof: Assume to the contrary that $\#S = \#\mathcal{P}(S)$. Then, there is a bijection f from S to $\mathcal{P}(S)$. Define $X \subset S$ according to the rule that $s \in X$ if and only if $s \notin f(s)$. ...

    (*Remark*: This is a harder question than something which would appear on the midterm.)

12. Prove that for any sets A and B, $A \times B \approx B \times A$.

13. (a) State the Pigeonhole principle.

    (b) Explain why there are two people in our class whose phone numbers end with the same digit.

14. Prove that if $A \subseteq B$ then $\#A \leq \#B$.