Math 308: Practice for Midterm 2

Prof. Hooper

Disclaimer. This test is just a recommendation of things to study and problems to work on. You may be asked about things that do not appear here. You should practice doing problems from the book in addition to the problems included in this sheet.

Covered material. The midterm will cover §2.5-2.10.

Definitions. You will be asked to define several terms on the test. **These terms all have one definition, as given in the book. You are expected to know this definition.** The following is a list of terms which might appear. (Others might appear as well).

set difference, complement, universal set, union, intersection, disjoint, associative laws (for sets), distributive laws, De Morgans laws, indexes, index set, indexed set, union/intersection of an indexed family of sets, disjoint (for indexed family of sets), pairwise disjoint, power set, ordered pair (know the "quasi-definition"), first and second coordinates, Cartesian product, partition, blocks of a partition, relation, reflexive, symmetric, transitive, equivalence relation, equivalent, equivalence class, partition induced by an equivalence relation, equivalence relation induced by a partition, finer, well-ordering principle, basis step, induction step, induction hypothesis, multiple.

Problems. These problems are intended to prepare you for the midterm, though not all of them are problems I would put on a midterm. Also consider the exercises in the book (both assigned and unassigned).

1. Let $A = \{2, 3\}$ and $B = \{3, 5\}$. Answer the following questions.

   (a) What is $A \cup B$?

   Solution: We have $A \cup B = \{2, 3, 5\}$.

   (b) Find $\mathcal{P}(A \cup B) - \big(\mathcal{P}(A) \cup \mathcal{P}(B)\big)$.

   Solution: The set $\mathcal{P}(A \cup B) - \big(\mathcal{P}(A) \cup \mathcal{P}(B)\big)$ consists of all subsets of $A \cup B$ which are not subsets of A and not subsets of B. Therefore,

   $$\mathcal{P}(A \cup B) - \big(\mathcal{P}(A) \cup \mathcal{P}(B)\big) = \big\{\{2, 5\}, \{2, 3, 5\}\big\}.$$

   (c) Describe the set $\mathbb{N} - A$ as $\{x \in \mathbb{N} \, : \, p(x)\}$ where $p(x)$ is some condition on x.

   Solution: The set $\mathbb{N} - A$ is also $\{1\} \cup \{4, 5, 6, \ldots\}$. So we have

   $$\mathbb{N} - A = \{x \in \mathbb{N} \, : \, x = 1 \text{ or } x \geq 4\}.$$

2. Let A and B be subsets of some set U. Suppose that $A \setminus B = \{5, 8\}$, $A \cap B = \{1\}$, $U \setminus B = \{4, 5, 8\}$, and $A \cup B = \{1, 3, 5, 7, 8\}$. Determine U, A and B.

Solution: The set $A \cup B$ contains everything in either A or B, and $A \setminus B$ contains all elements of A which are not in B. So,

$$B = (A \cup B) \setminus (A \setminus B) = \{1, 3, 5, 7, 8\} \setminus \{5, 8\} = \{1, 3, 7\}.$$

Since $U \setminus B = \{4, 5, 8\}$, we must have

$$U = B \cup (U \setminus B) = \{1, 3, 7\} \cup \{4, 5, 8\} = \{1, 3, 4, 5, 7, 8\}.$$

Also, $A \setminus B$ consists of elements which are in A but not B, and $A \cap B$ consists of elements which are in both A and B. So,

$$A = (A \cap B) \cup (A \setminus B) = \{1\} \cup \{5, 8\} = \{1, 5, 8\}.$$

3. Let A, B and C be sets. Prove that

$$A - (B \cap C) = (A - B) \cup (A - C).$$

Solution: We will begin by showing that $A-(B\cap C) \subseteq (A-B)\cup(A-C)$. Suppose that $x \in A-(B\cap C)$. Then $x \in A$ and $x \notin B\cap C$. So, $x \notin B$ or $x \notin C$. By possibly switching the names of the sets B and C, we can assume without loss of generality that $x \notin B$. Since $x \in A$ and $x \notin B$, we know $x \in A - B$. Therefore, we also have $x \in (A - B) \cup (A - C)$.

Now we will show that $(A-B)\cup(A-C) \subseteq A-(B\cap C)$. Let $x \in (A-B)\cup(A-C)$. Then $x \in A-B$ or $x \in A-C$. Without loss of generality, we may assume $x \in A-B$. Then $x \in A$ and $x \notin B$. Therefore $x \notin B \cap C$. Thus we have $x \in A - (B \cap C)$.

4. This problem concerns the quantified statement S:

For all sets A and B, if $A \cap C = B \cap C$ for every set C, then $A = B$.

(a) State the negation of the statement S in a complete sentence.

Solution: There exist sets A and B so that $A \cap C = B \cap C$ for every set C and $A \neq B$.

(b) Prove or disprove the statement S.

Solution: The statement is true. Let A and B be sets for which $A \cap C = B \cap C$ for every set C. Then in particular,

$$A \cap (A \cup B) = B \cap (A \cup B).$$

Because $A \subseteq A \cup B$, we know that $A \cap (A \cup B) = A$. Similarly, $B \cap (A \cup B) = B$. Thus, $A = B$.

5. Let $A = \{1, 2, 3\}$, $B = \{2, 3, 5\}$ and $C = \{3, 4, 5\}$.
   (a) What is $C \setminus (A \cup B)$?

   Solution: $C \setminus (A \cup B) = \{4\}$.

   (b) What is $B \cap C$?

   Solution: $B \cap C = \{3, 5\}$.

   (c) What is $\mathcal{P}(A \cap B)$?

   Solution: Since $A \cap B = \{2, 3\}$, we have $\mathcal{P}(A \cap B) = \big\{\emptyset, \{2\}, \{3\}, \{2, 3\}\big\}$.

   (d) What is the following set?

$$D = \{b \in B \ : \ \text{there exists } a \in A \text{ and } c \in C \text{ such that } b = a + c\}.$$

   Solution: Observe that if $a \in A$ and $c \in C$, then $a \geq 1$ and $c \geq 3$. So, $a + c \geq 4$. The only $b \in B$ which satisfies $b \geq 4$ is $b = 5$. Observe that $1 \in A$ and $4 \in C$, so $5 = 1 + 4 \in D$. So, we have $D = \{5\}$.

6. Prove that $7^n - 1$ is a multiple of 3 for every non-negative integer n. (*Hint*: Try induction.)

Solution: We will prove this by induction. For our base case, consider when $n = 0$. We have $7^0 - 1 = 0$ which is a multiple of 3 since $0 = 3 \cdot 0$.

Now assume the inductive hypothesis that $7^n - 1$ is a multiple of 3. We will prove that $7^{n+1} - 1$ is a multiple of 3. By hypothesis, we see that $7^n = 3x + 1$ for some integer x. Then,

$$7^{n+1} - 1 = 7(7^n) - 1 = 7(3x + 1) - 1 = 21x + 6 = 3(7x + 2).$$

Since $7x + 2$ is an integer, $7^{n+1} - 1$ is a multiple of 3.

By the Principle of Mathematical Induction, $7^n - 1$ is a multiple of 3 for every integer $n \geq 0$.

7. Let x be a real number with $x \neq 1$. Use induction to prove that for all non-negative integers n,

$$\sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}.$$

(Note that the sum expands to $1 + x + x^2 + \ldots + x^n$. This is a *geometric series*.)

Solution: Consider the base case of $n = 0$. In this case,

$$\sum_{k=0}^{0} x^k = x^0 = 1 \quad \text{and} \quad \frac{1 - x^{0+1}}{1 - x} = 1.$$

So, the two sides of the equation are equal.

Now suppose the equation is true for $n = m$. We will show it is true for $n = m + 1$. So, by hypothesis, we have

$$\sum_{k=0}^{m} x^k = \frac{1 - x^{m+1}}{1 - x}.$$

Considering the sum for $n = m + 1$, we have:

$$\sum_{k=0}^{m+1} x^k = x^{m+1} + \sum_{k=0}^{m} x^k = x^{m+1} + \frac{1 - x^{m+1}}{1 - x} = \frac{(1 - x)x^{m+1} + (1 - x^{m+1})}{1 - x} = \frac{1 - x^{m+2}}{1 - x}.$$

So, the equation holds for $n = m + 1$.

By the Principle of Mathematical Induction, the statement is true for all $n \geq 0$.

8. The notion of "divides" is a relation on $\mathbb{N}$. Recall that for $a, b \in \mathbb{N}$ that a divides b if there is an integer c so that $b = ac$. We denote this relation by $|$, so "$a|b$" means "a divides b."

   (a) Prove or disprove that $|$ is a reflexive relation on $\mathbb{N}$.

   Solution: The relation is reflexive.
   Proof. Let $a \in \mathbb{N}$. Then $a = a \cdot 1$, so $a|a$.

   (b) Prove or disprove that $|$ is a symmetric relation on $\mathbb{N}$.

Solution: The relation is not symmetric.
Proof. We will give a counterexample. Let $a = 1$ and $b = 2$. Then $a|b$, because $b = 2a$. But, $b \not| a$, because 2 only divides even integers (or because $\frac{1}{2} \notin \mathbb{Z}$).

(c) Prove or disprove that | is a transitive relation on $\mathbb{N}$.

Solution: The relation is transitive.
Proof. Suppose that $a|b$ and $b|c$. Then there are integers x and y so that $b = ax$ and $c = by$. Then, $c = by = axy$. Since $xy \in \mathbb{Z}$, we see that $a|c$.

(d) Prove or disprove that | is an equivalence relation on $\mathbb{N}$.

Solution: The relation is not an equivalence relation because it is not symmetric.

9. Consider the relation on $\mathbb{R}$ defined by $x \sim y$ if $y - x \in \mathbb{Z}$.

(a) Show that $\sim$ is an equivalence relation on $\mathbb{R}$.

Solution: We will show $\sim$ is reflexive. Let $x \in \mathbb{R}$. Then $x - x = 0 \in \mathbb{Z}$ so $x \sim x$.
We will show $\sim$ is symmetric. Let $x, y \in \mathbb{R}$ and assume $x \sim y$. Then $y - x \in \mathbb{Z}$. Since negation preserves $\mathbb{Z}$, we know $x - y = -(y - x) \in \mathbb{Z}$. We have shown $y \sim x$.
We will show $\sim$ is transitive. Let $x, y \in \mathbb{R}$ and assume $x \sim y$ and $y \sim z$. Then $y - x \in \mathbb{Z}$ and $z - y \in \mathbb{Z}$. By adding the two integers, we see

$$z - x = (z - y) + (y - x) \in \mathbb{Z}.$$

Thus $z \sim x$.
Since $\sim$ is reflexive, symmetric, and transitive, it follows that $\sim$ is an equivalence relation.

(b) Give an explicit description of the equivalence class $[\frac{1}{2}]$ as a subset of $\mathbb{R}$.

Solution: Observe that $\frac{1}{2} \sim y$ if and only if $y - \frac{1}{2} \in \mathbb{Z}$. So $\frac{1}{2} \sim y$ if and only if there is an $n \in \mathbb{Z}$ so that $y = \frac{1}{2} + n$. So,

$$[\frac{1}{2}] = \{\frac{1}{2} + n \ : \ n \in \mathbb{Z}\}.$$

10. Prove that if A is any set, then $A \times \emptyset = \emptyset$.

Solution: Let A be any set. We'll show $A \times \emptyset = \emptyset$. Suppose not. Then $A \times \emptyset \neq \emptyset$. This means that there is an $x \in A \times \emptyset$. Since x is an element of a Cartesian product, we know x is a pair $x = (x_1, x_2)$. Since $(x_1, x_2) \in A \times \emptyset$ we know that $x_1 \in A$ and $x_2 \in \emptyset$. But $x_2 \in \emptyset$ is always false, so this contradicts our assumption that $A \times \emptyset \neq \emptyset$. We conclude that $A \times \emptyset = \emptyset$.

11. Consider the following statement:
For any two sets A and B, $P(A \times B) = P(A) \times P(B)$.
Is this statement true or false. Prove that your answer is correct.

Solution: It is false. For example if $A = B = \mathbb{N}$, then $X = \{(1,1),(2,2)\} \in P(A \times B)$ since $\{(1,1),(2,2)\} \subset \mathbb{N} \times \mathbb{N}$. On the other hand $X \notin P(A) \times P(B)$. If X were in $P(A) \times P(B)$, then there would be a $C \in P(A)$ and a $D \in P(B)$ so that $X = (C, D)$. But then X would be a pair of sets not a set of pairs.

12. Let $\{A_i\}_{i \in I}$ be an indexed family of sets.
   (a) Complete the following definition: The union $\bigcup_{i \in I} A_i$ is the set ...

   Solution: of x so that there is an $i \in I$ such that $x \in A_i$.

   (b) Complete the following definition: The intersection $\bigcup_{i \in I} A_i$ is the set ...

   Solution: of x so that for all $i \in I$ we have $x \in A_i$.

   (c) Prove that for any set A and any family of sets $\{B_i\}_{i \in I}$ with $I \neq \emptyset$, we have

$$A - \bigcup_{i \in I} B_i = \bigcap_{i \in I} (A - B_i).$$

   Solution: Let $X = A - \bigcup_{i \in I} B_i$ and $Y = \bigcap_{i \in I}(A - B_i)$.
   First we will show $x \in X$ implies $x \in Y$. Let $x \in X$. Then $x \in A - \bigcup_{i \in I} B_i$, which means that $x \in A$ and $x \notin \bigcup_{i \in I} B_i$. This means that there is no $i \in I$ so that $x \in B_i$, or equivalently, for all i, $x \notin B_i$. Then for all i, we see that $x \in A - B_i$. This tells us that $x \in \bigcap_{i \in I}(A - B_i) = Y$.
   Now we'll show that $y \in Y$ implies $y \in X$. Let $y \in Y = \bigcap_{i \in I}(A - B_i)$. Then for each $i \in I$, $y \in A - B_i$. So, for each $i \in I$, we see that $y \in A$ and $y \notin B_i$. As long as $I \neq \emptyset$ (which is required for an indexed collection of sets), we see that $y \in A$. Since $y \notin B_i$ for all $i \in I$, we see that $y \notin \bigcup_{i \in I} B_i$. Thus $y \in A - \bigcup_{i \in I} B_i = X$.

13. (a) Complete the following definition:
A partition Π of a non-empty set S is ...

Solution: A *partition* Π of a non-empty set S is a family $\Pi = \{A_i\}_{i\in I}$ of non-empty subsets of S satisfying

$$\bigcup_{i\in I} A_i = S \quad \text{and} \quad A_i \cap A_j = \emptyset \text{ whenever } i \neq j.$$

(b) Suppose $\mathcal{A} = \{A_i\}_{i\in I}$ and $\mathcal{Q} = \{B_j\}_{j\in J}$ are both partitions of a set S. For $i \in I$ and $j \in J$ define $C_{i,j} = A_i \cap B_j$. Let $K \subseteq I \times J$ be the set of pairs (i, j) so that $C_{i,j} \neq \emptyset$. Prove that the family of sets

$$\mathcal{R} = \{C_{i,j}\}_{(i,j)\in K}$$

is a partition.

Solution: Observe that because each $C_{i,j} = A_i \cap B_j$, each $A_i \subseteq S$, and each $B_j \subseteq S$, we know that each $C_{i,j} \subseteq S$. This guarantees that $\mathcal{R} = \{C_{i,j}\}_{(i,j)\in K}$ is a family of subsets of S.

Now we will show that $\bigcup_{(i,j)\in K} C_{i,j} = S$. Since $\mathcal{R}$ is a family of subsets of S, we know that $\bigcup_{(i,j)\in K} C_{i,j} \subseteq S$. We will now show that $S \subseteq \bigcup_{(i,j)\in K} C_{i,j}$ by proving that $s \in S$ implies that $s \in \bigcup_{(i,j)\in K} C_{i,j}$. Let $s \in S$ be arbitrary. Then since $\mathcal{A}$ is a partition of S, there is an $i \in I$ so that $s \in A_i$. Since $\mathcal{B}$ is a partition of S, there is a $j \in J$ so that $s \in B_j$. Then $s \in A_i \cap B_j = C_{i,j}$. Since $C_{i,j} \neq \emptyset$, we know that $(i, j) \in K$. We have shown that s lies in $C_{i,j}$ for some $(i, j) \in K$, so we know that $s \in \bigcup_{(i,j)\in K} C_{i,j}$. This completes the proof that $S \subseteq \bigcup_{(i,j)\in K} C_{i,j}$ and since we also know $\bigcup_{(i,j)\in K} C_{i,j} \subseteq S$, we see that $\bigcup_{(i,j)\in K} C_{i,j} = S$.

Now we need to prove that for all $(i, j) \in K$ and all $(i', j') \in K$, $(i, j) \neq (i', j')$ implies $C_{i,j} \cap C_{i',j'} = \emptyset$. We'll prove this by contrapositive. Suppose that $C_{i,j} \cap C_{i',j'} \neq \emptyset$. Then there is an $x \in C_{i,j} \cap C_{i',j'}$. Then $x \in C_{i,j}$ and $x \in C_{i',j'}$. Then by definition of the C_* sets, $x \in A_i \cap B_j$ and $x \in A_{i'} \cap B_{i'}$. So x is in each of the four sets A_i, $A_{i'}$, B_j and $B_{j'}$. This shows that $A_i \cap A_{i'} \neq \emptyset$ and $B_j \cap B_{j'} \neq \emptyset$. Since $\mathcal{A}$ and $\mathcal{B}$ are partitions it follows that $i = i'$ and $j = j'$. We conclude that $(i, j) = (i', j')$ which completes our proof by contrapositive.

14. (a) Let S be a set. The *power set* of S is ...

Solution: The set of all subsets of S.

(b) Let A and B be sets. Prove that $A \subseteq B$ if and only if $P(A) \subseteq P(B)$.

Solution: First we'll show that $A \subseteq B$ implies that $P(A) \subseteq P(B)$. Assume $A \subseteq B$. We'll show $P(A) \subseteq P(B)$. Let $X \in P(A)$. Then by definition of the power set $X \subseteq A$. Since $X \subset A$ and $A \subseteq B$, we know that $X \subseteq B$. This means that $X \in P(B)$ which proves $P(A) \subseteq P(B)$.

Now we'll show that $P(A) \subseteq P(B)$ implies that $A \subseteq B$ by contrapositive. Assume that $A \nsubseteq B$. Then there is an $a \in A$ such that $a \notin B$. Consider the set $Y = \{a\}$. Then since $a \in A$ we know $Y \subseteq A$ and so $Y \in P(A)$. On the other hand $a \notin B$ so we know $Y \nsubseteq B$ and so $Y \notin P(B)$. We have showed that $P(A) \nsubseteq P(B)$ because $P(A)$ contains the element Y which is not in $P(B)$.

15. Show that if A, B and C are sets, then $(A \cup B) \times C = (A \times C) \cup (B \times C)$.

Solution: First suppose $x \in (A \cup B) \times C$. We'll show $x \in (A \times C) \cup (B \times C)$. Since $x \in (A \cup B) \times C$, we know $x = (x_1, x_2)$ with $x_1 \in A \cup B$ and $x_2 \in C$. Since $x_1 \in A \cup B$, we know $x_1 \in A$ or $x_1 \in B$. We'll break into cases accordingly. First assume $x_1 \in A$. Then $(x_1, x_2) \in A \times C$, so that $(x_1, x_2) \in (A \times C) \cup (B \times C)$. Now suppose $x_2 \in B$. Then $(x_1, x_2) \in B \times C$, which means that again $(x_1, x_2) \in (A \times C) \cup (B \times C)$. In any case, we've shown $x \in (A \times C) \cup (B \times C)$.

Now suppose that $y \in (A \times C) \cup (B \times C)$. Then $y \in A \times C$ or $y \in B \times C$. We'll break into cases. If $y \in A \times C$ then $y = (y_1, y_2)$ with $y_1 \in A$ and $y_2 \in C$. Then, $y_1 \in A \cup B$ and $y_2 \in C$ so that $y = (y_1, y_2) \in (A \cup B) \times C$. For the second possible case, suppose $y \in B \times C$. Then $y = (y_1, y_2)$ with $y_1 \in B$ and $y_2 \in C$. Then $y_1 \in A \cup B$ so $y \in (A \cup B) \times C$. In both cases we have shown $y \in (A \cup B) \times C$.