Math 323: Practice for Midterm 2

Prof. Hooper

Disclaimer. This test is just a recommendation of things to study and problems to work on. You may be asked about things that do not appear here. You should practice doing problems from the book in addition to the problems included in this sheet.

Covered material. The midterm will cover §2.5-2.10.

Definitions. You will be asked to define several terms on the test. **These terms all have one definition, as given in the book. You are expected to know this definition.** The following is a list of terms which might appear. (Others might appear as well).

set difference, complement, universal set, union, intersection, disjoint, associative laws (for sets), distributive laws, De Morgans laws, indexes, index set, indexed set, union/intersection of an indexed family of sets, disjoint (for indexed family of sets), pairwise disjoint, power set, ordered pair (know the "quasi-definition"), first and second coordinates, Cartesian product, partition, blocks of a partition, relation, reflexive, symmetric, transitive, equivalence relation, equivalent, equivalence class, partition induced by an equivalence relation, equivalence relation induced by a partition, finer, well-ordering principle, basis step, induction step, induction hypothesis, multiple.

Problems. These problems are intended to prepare you for the midterm, though not all of them are problems I would put on a midterm. Also consider the exercises in the book (both assigned and unassigned).

1. Let $A = \{2, 3\}$ and $B = \{3, 5\}$. Answer the following questions.
   (a) What is $A \cup B$?
   (b) Find $\mathcal{P}(A \cup B) - \big(\mathcal{P}(A) \cup \mathcal{P}(B)\big)$.
   (c) Describe the set $\mathbb{N} - A$ as $\{x \in \mathbb{N} \ : \ p(x)\}$ where $p(x)$ is some condition on x.

2. Let A and B be subsets of some set U. Suppose that $A \setminus B = \{5, 8\}$, $A \cap B = \{1\}$, $U \setminus B = \{4, 5, 8\}$, and $A \cup B = \{1, 3, 5, 7, 8\}$. Determine U, A and B.

3. Let A, B and C be sets. Prove that

$$A - (B \cap C) = (A - B) \cup (A - C).$$

4. This problem concerns the quantified statement S:

For all sets A and B, if $A \cap C = B \cap C$ for every set C, then $A = B$.

   (a) State the negation of the statement S in a complete sentence.
   (b) Prove or disprove the statement S.

5. Let $A = \{1, 2, 3\}$, $B = \{2, 3, 5\}$ and $C = \{3, 4, 5\}$.
   (a) What is $C \setminus (A \cup B)$?
   (b) What is $B \cap C$?
   (c) What is $\mathcal{P}(A \cap B)$?
   (d) What is the following set?

$$D = \{b \in B \ : \ \text{there exists } a \in A \text{ and } c \in C \text{ such that } b = a + c\}.$$

6. Prove that $7^n - 1$ is a multiple of 3 for every non-negative integer n. (*Hint*: Try induction.)

7. Let x be a real number with $x \neq 1$. Use induction to prove that for all non-negative integers n,

$$\sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}.$$

   (Note that the sum expands to $1 + x + x^2 + \ldots + x^n$. This is a *geometric series.*)

8. The notion of "divides" is a relation on $\mathbb{N}$. Recall that for $a, b \in \mathbb{N}$ that a divides b if there is an integer c so that $b = ac$. We denote this relation by $|$, so "$a|b$" means "a divides b."
   (a) Prove or disprove that $|$ is a reflexive relation on $\mathbb{N}$.
   (b) Prove or disprove that $|$ is a symmetric relation on $\mathbb{N}$.
   (c) Prove or disprove that $|$ is a transitive relation on $\mathbb{N}$.
   (d) Prove or disprove that $|$ is an equivalence relation on $\mathbb{N}$.

9. Consider the relation on $\mathbb{R}$ defined by $x \sim y$ if $y - x \in \mathbb{Z}$.
   (a) Show that $\sim$ is an equivalence relation on $\mathbb{R}$.
   (b) Give an explicit description of the equivalence class $[\frac{1}{2}]$ as a subset of $\mathbb{R}$.

10. Prove that if A is any set, then $A \times \emptyset = \emptyset$.

11. Consider the following statement:
    For any two sets A and B, $P(A \times B) = P(A) \times P(B)$.
    Is this statement true or false. Prove that your answer is correct.

12. Let $\{A_i\}_{i \in I}$ be an indexed family of sets.
    (a) Complete the following definition: The union $\bigcup_{i \in I} A_i$ is the set ...
    (b) Complete the following definition: The intersection $\bigcup_{i \in I} A_i$ is the set ...
    (c) Prove that for any set A and any family of sets $\{B_i\}_{i \in I}$ we have

$$A - \bigcup_{i \in I} B_i = \bigcap_{i \in I} (A - B_i).$$

13. (a) Complete the following definition:
    A partition Π of a non-empty set S is ...

    (b) Suppose $\mathcal{A} = \{A_i\}_{i \in I}$ and $\mathcal{Q} = \{B_j\}_{j \in J}$ are both partitions of a set S. For $i \in I$ and $j \in J$ define $C_{i,j} = A_i \cap B_j$. Let $K \subseteq I \times J$ be the set of pairs (i, j) so that $C_{i,j} \neq \emptyset$. Prove that the family of sets

$$\mathcal{R} = \{C_{i,j}\}_{(i,j) \in K}$$

    is a partition.

14. (a) Let S be a set. The *power set* of S is ...

    (b) Let A and B be sets. Prove that $A \subseteq B$ if and only if $P(A) \subseteq P(B)$.

15. Show that if A, B and C are sets, then $(A \cup B) \times C = (A \times C) \cup (B \times C)$.