Math 323: Practice for Midterm 1

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Disclaimer. This test is just a recommendation of things to study and problems to work on. You may be asked about things that do not appear here. You should practice doing problems from the book in addition to the problems included in this sheet.

Covered material. The midterm will cover §1.1-1.5 and §2.1-2.4.

Definitions. You will be asked to define several terms on the test. **These terms all have one definition, as given in the book. You are expected to know this definition.** The following is a list of terms which might appear. (Others might appear as well).

axioms, proposition, truth value, prime number, truth functional, connective, truth table, conjunction, statement forms (or compound statement), statement letters, disjunction, inclusive or, exclusive or, conditional, implication, hypothesis, conclusion, converse, tautology, modus ponens, direct proof, proof by contradiction, contrapositive, logically equivalent, logically equivalence, De Morgans laws, contradiction, set, member, element, equality of sets, natural numbers $\mathbb{N}$*, integers* $\mathbb{Z}$*, rational numbers* $\mathbb{Q}$*, real numbers* $\mathbb{R}$*, irrational numbers (i.e., real numbers that are not rational), even integer, odd integer, empty set, variable, existential quantifier, universal quantifier, universal set, subset, proper subset*

Problems. These problems are intended to prepare you for the midterm, though not all of them are problems I would put on a midterm. Also consider the exercises in the book (both assigned and unassigned).

1. Show that the following statement forms are logically equivalent.

$$S_1 : (P \vee Q) \Rightarrow R.$$

$$S_2 : \big((\sim P) \wedge (\sim Q)\big) \vee R.$$

Solution: We prove that these statements are logically equivalent using a truth table:

P	Q	R	$P \vee Q$	$(P \vee Q) \Rightarrow R$	$\sim P$	$\sim Q$	$(\sim P) \wedge (\sim Q)$	$\big((\sim P) \wedge (\sim Q)\big) \vee R$
T	T	T	T	T	F	F	F	T
T	T	F	T	F	F	F	F	F
T	F	T	T	T	F	T	F	T
T	F	F	T	F	F	T	F	F
F	T	T	T	T	T	F	F	T
F	T	F	T	F	T	F	F	F
F	F	T	F	T	T	T	T	T
F	F	F	F	T	T	T	T	T

The columns for the statements $(P \vee Q) \Rightarrow R$ and $\big((\sim P) \wedge (\sim Q)\big) \vee R$ are identical, so these statements are logically equivalent.

2. Let $x, y \in \mathbb{Z}$. Prove that $(x+1)(2x+y)$ is odd if and only if x is even and y is odd.

Solution: First, we will show that if x is even and y is odd, then $(x+1)(2x+y)$ is odd. Suppose x is even and y is odd. Then we have $x = 2a$ and $y = 2b+1$ for some $a, b \in \mathbb{Z}$. So,

$$(x+1)(2x+y) = (2a+1)(4a+2b+1) = 2\big(a(4a+2b+1) + (2a+b)(2a+1)\big) + 1.$$

Since $a(4a+2b+1) + (2a+b)(2a+1) \in \mathbb{Z}$, we know $(x+1)(2x+y)$ is odd.

We will now show that if x is odd or y is even, then $(x+1)(2x+y)$ is even We break into cases.
Case 1: Suppose that x is odd. Then $x = 2a+1$ for some $a \in \mathbb{Z}$. So,

$$(x+1)(2x+y) = (2a+2)(2x+y) = 2(a+1)(2x+y).$$

Since $(a+1)(2x+y) \in \mathbb{Z}$, we know $(x+1)(2x+y)$ is even.
Case 2: Suppose that y is even. Then $y = 2b$ for some $b \in \mathbb{Z}$. So,

$$(x+1)(2x+y) = (x+1)(2x+2b) = 2(x+1)(x+b).$$

Since $(x+1)(x+b) \in \mathbb{Z}$, we know $(x+1)(2x+y)$ is even.

3. Let $x, y, z, \in \mathbb{R}$. Prove that if $6x + 6y + 4 < 13z$, then $2x < 3z$ or $3y + 2 < 2z$.

Solution: We will prove this by contrapositive. Suppose $2x \geq 3z$ and $3y + 2 \geq 2z$. Then

$$6x + 6y + 4 = 3(2x) + 2(3y+2) \geq 3(3z) + 2(2z) = 13z.$$

Thus, $6x + 6y + 4 \geq 13z$ as desired.

4. Answer the following questions. Express your answer in a complete sentence. Change quantifiers as appropriate.

   (a) State the negation of the following statement:
   There is a real number z so that $z - z^2 > 1$.

   Solution: For every real number z, $z - z^2 \leq 1$.

   (b) State the negation of the following statement:
   For each $n \in \mathbb{N}$, if $n! > 4^n$, then $n! + 1$ is a prime number.

Solution: There is an $n \in \mathbb{N}$ so that $n! > 4^n$ and $n! + 1$ is not prime.

(c) Let $m \in \mathbb{Z}$. State the contrapositive of the following implication:
If m^2 is even, then $m^3 - 1$ is odd.

Solution: If $m^3 - 1$ is even, then m^2 is odd.

(d) Let $x \in \mathbb{R}$. State the contrapositive of the following implication:
If $x(x-2) = 0$, then $x = 0$ or $x = 2$.

Solution: If $x \neq 0$ and $x \neq 2$, then $x(x-2) \neq 0$.

5. For statements P and Q, show that $\big((\sim Q) \wedge (P \Rightarrow Q)\big) \Rightarrow (\sim P)$ is a tautology.

Solution: The following is a truth table which computes the truth value of the statement given the truth value of P and Q. This statement is a tautology, because its only possible value is true.

P	Q	$\sim Q$	$P \Rightarrow Q$	$(\sim Q) \wedge (P \Rightarrow Q)$	$\sim P$	$\big((\sim Q) \wedge (P \Rightarrow Q)\big) \Rightarrow (\sim P)$
T	T	F	T	F	F	T
T	F	T	F	F	F	T
F	T	F	T	F	T	T
F	F	T	T	T	T	T

6. Prove the following statement is true:
For all integers a and b, if a is odd and b is even, then $\frac{ab+3b}{4}$ is an integer.

Solution: Assume that a is an odd integer and that b is an even integer. Since a is odd, there is an integer m so that $a = 2m+1$. Since b is even, there is an integer n so that $b = 2n$. Then we compute:

$$ab + 3b = (2m+1)(2n) + 3(2n) = 4mn + 2n + 6n = 4mn + 8n = 4(mn + 2n).$$

So, we see that $\frac{ab+3b}{4} = mn + 2n$, which is an integer.

7. Prove the following statement is true:
For all $n \in \mathbb{Z}$, if $n(n+3)$ is odd, then $n^3 - 2n \geq 0$.

Solution: We will show that this statement is vacuously true by showing that $n(n+3)$ is even for every $n \in \mathbb{Z}$.

To prove this, we break into cases. First, suppose that n is even. Then $n = 2a$ for some $a \in \mathbb{Z}$. So, we have

$$n(n+3) = 2a(n+3) = 2(an+3a).$$

Since $an+3a$ is an integer, $n(n+3)$ is even.

Now suppose that n is odd. Then $n = 2b+1$ for some $b \in \mathbb{Z}$. In this case, we have

$$n(n+3) = n(2b+1+3) = n(2b+4) = 2n(b+2).$$

Since $n(b+2)$ is an integer, $n(n+3)$ is even.

We have proved that $n(n+3)$ for all $n \in \mathbb{N}$.

8. Prove that if x is a real number so that $1 < x \leq 3$, then $\frac{8}{x^2-1} \geq 1$.
(*Hint:* Observe that $x^2 - 1 = (x+1)(x-1)$.)

Solution:

Observations before proof. Observe that $x^2 - 1 > 0$ when $x > 1$. So by multiplying by $x^2 - 1$, the inequality $\frac{8}{x^2-1} > 1$ is equivalent to $8 > x^2 - 1 = (x+1)(x-1)$. To get this, we observe that both $x+1$ and $x-1$ are positive and $x+1 \leq 4$ and $x-1 \leq 2$.

Formal Proof. We will give a direct proof. Suppose that $1 < x \leq 3$. Since $x > 1$, both $x+1$ and $x-1$ are positive, and so their product $x^2-1 = (x+1)(x-1)$ is also positive. Since $x \leq 3$, we know that $x+1 \leq 4$ and $x-1 \leq 2$. Then,

$$x^2 - 1 = (x+1)(x-1) \leq 4(x-1) \leq 4 \cdot 2 = 8.$$

Since $x^2 - 1 > 0$, the inequality $x^2 - 1 \leq 8$ is equivalent to $1 \leq \frac{8}{x^2-1}$.

9. Consider the following statement:

P : For all non-zero rational numbers x and all irrational numbers y, the product xy is irrational.

(a) Write the negation of P in a sentence. (You must appropriately alter the quantifiers.)

Solution: There exists a non-zero rational number x and an irrational number y so that xy is rational.

(b) Prove the statement P is true. (*Hint:* Try a proof by contradiction.)

Solution: Suppose P is false. Then there exists a non-zero rational numbers x and an irrational number y so that xy is rational. Let $z = xy$ be that rational. Since x is non-zero, we have $y = z/x$. But, x and z are both rational, so their quotient, y, is also rational. But then y is both rational and irrational, which is a contradiction.

10. In each of the following parts, a universal set is specified, and then a quantified statement is expressed. Write negations of these quantified statements.

(a) Let $S \subset \mathbb{R}$. For every $x \in S$, $x^4 \geq x^2$.

Solution: There is an $x \in S$ so that $x^4 < x^2$.

(b) Let $S \subset \mathbb{R}$. There exist elements $s \in S$ and $t \in S$ so that $st = 1$.

Solution: For any two elements $s \in S$ and $t \in S$ we have $st \neq 1$.

(c) Let A be a set of non-negative real numbers. For any $a \in A$, if $\sqrt{a}$ is a rational number, then $\sqrt{a}$ is an integer.

Solution: Solution 1: There is an $a \in A$ so that $\sqrt{a}$ is rational and $\sqrt{a}$ is not an integer.
Solution 2: There is an $a \in A$ so that $\sqrt{a} \in \mathbb{Q} \setminus \mathbb{N}$.

(d) Let $X \subset \mathbb{R}$ and $m \in \mathbb{R}$. We have $-m \leq x \leq m$ for any $x \in X$.

Solution: There is an $x \in X$ such that $x > m$ or $x < -m$.
Remark: The statement $-m \leq x \leq m$ means $-m \leq x$ and $x \leq m$.

(e) Let $X \subset \mathbb{R}$. For every non-empty subset Y of X, there is an $m \in Y$ with the property that $m \leq y$ for any $y \in Y$.

Solution: There is a non-empty subset Y of X such that for any $m \in Y$, there is a $y \in Y$ such that $y < m$.
Remark: If X satisfies the statement, it has the property that any subset of X has a minimal element. This is true for $X = \mathbb{N}$ for example and is important for induction which we will discuss soon.

11. Consider the implication

$$P: \text{ If } x^2 = 4, \text{ then } x^2 = x + 6.$$

For which $x \in \mathbb{R}$ is P true? For which $x \in \mathbb{R}$ is P false?

Solution: An implication is only false when the hypothesis is true and the conclusion is false. The statement $x^2 = 4$ is true precisely when $x = 2$ or $x = -2$. Since $x^2 = x+6$ is equivalent to $x^2 - x - 6 = 0$ and we can write this as $(x+2)(x-3) = 0$, we see that the statement $x^2 = x + 6$ is false unless $x = -2$ or $x = 3$ (in which cases the statement is true). Thus, the implication P is false only when $x = 2$. The statement P is true when $x \neq 2$.

12. Describe each of the following sets in the format $\{x \in \mathbb{N} : \ P(x)\}$.

    (a) $\{3, 6, 9, 12, 15, \ldots\}$.

    Solution:
    $$\{x \in \mathbb{N} : \ \exists k \in \mathbb{N}, x = 3k\}.$$

    (b) $\{1, 10, 100, 1000, 10000, \ldots\}$.

    Solution:
    $$\{x \in \mathbb{N} : \ \exists k \in \mathbb{N}, x = 10^k\}$$
    or
    $$\{x \in \mathbb{N} : \ \text{the digits of } x \text{ sum to one}\}.$$

    (c) $\{1, 5, 437\}$.

    Solution:
    $$\{x \in \mathbb{N} : \ x = 1 \text{ or } x = 5 \text{ or } x = 437\}$$
    or
    $$\{x \in \mathbb{N} : \ (x-1)(x-5)(x-437) = 0\}.$$

13. Express the following statements using quantifiers. Include any implicit universal quantifiers.

    (a) Every positive integer is the sum of two squares of integers.

    Solution:
    $$\forall x \in \mathbb{Z}, \exists a \in \mathbb{Z}, \exists b \in \mathbb{Z}, x > 0 \implies x = a^2 + b^2$$
    or
    $$\forall x \in \mathbb{Z}, x > 0 \implies (\exists a \in \mathbb{Z}, \exists b \in \mathbb{Z}, x = a^2 + b^2).$$

    (b) If x is a positive real number, then so is $\arctan x$.

Solution:

$$\forall x \in \mathbb{R}, x > 0 \implies \arctan x > 0.$$

(c) There is a smallest natural number.

Solution:

$$\exists n_1 \in \mathbb{N}, \forall n_2 \in \mathbb{N}, n_1 \leq n_2.$$

(d) If $b^2 - 4ac \geq 0$, then $ax^2 + bx + c$ has a real root.

Solution:

$$\forall a \in \mathbb{R}, \forall b \in \mathbb{R}, \forall c \in \mathbb{R}, b^2 - 4ac \geq 0 \implies (\exists x \in \mathbb{R}, ax^2 + bx + c = 0).$$

14. (a) Complete the following definition.
If A and B are sets, then $A \subseteq B$ if . . .

Solution: for every element x, $x \in A$ implies $x \in B$.

(b) Complete the following definition.
If A and B are sets, then $A = B$ if . . .

Solution: for every element x, $x \in A$ if and only if $x \in B$.

(c) Let $\emptyset$ be the empty set and A be another set. Prove that $\emptyset \subseteq A$.

Solution: To prove $\emptyset \subset A$, we need to show that for any element x, if $x \in \emptyset$ then $x \in A$. But, the empty set has no elements, so $x \in \emptyset$ is always false. This makes the implication vacuously true (i.e., $x \in \emptyset$ then $x \in A$ is always true because $x \in \emptyset$ is always false).

(d) Let A, B and C be sets. Prove that if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Solution: Assume $A \subseteq B$ and $B \subseteq C$. We will show $A \subseteq C$. Let x be an element. We will give a direct proof that $x \in A$ implies $x \in C$. Suppose $x \in A$. Then since $A \subseteq B$, we know $x \in B$. Since $x \in B$ and $B \subseteq C$, we also have $x \in C$.

(e) Prove that if A is a set and $A \subseteq \emptyset$, then $A = \emptyset$.

Solution: We will give a direct proof. Suppose A is a set and that $A \subseteq \emptyset$. We will show $A = \emptyset$, We must show two statements for every element x: (1) if $x \in A$ then $x \in \emptyset$, and (2) if $x \in \emptyset$ then $x \in A$.

Statement (2) is vacuously true since $x \in \emptyset$ is always false.

Statement (1) is true because $A \subseteq \emptyset$ which is equivalent to the statement that for all elements x, $x \in A$ implies $x \in \emptyset$.