Math 323: Practice for Midterm 1

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Disclaimer. This test is just a recommendation of things to study and problems to work on. You may be asked about things that do not appear here. You should practice doing problems from the book in addition to the problems included in this sheet.

Covered material. The midterm will cover §1.1-1.5 and §2.1-2.4.

Definitions. You will be asked to define several terms on the test. **These terms all have one definition, as given in the book. You are expected to know this definition.** The following is a list of terms which might appear. (Others might appear as well).

axioms, proposition, truth value, prime number, truth functional, connective, truth table, conjunction, statement forms (or compound statement), statement letters, disjunction, inclusive or, exclusive or, conditional, implication, hypothesis, conclusion, converse, tautology, modus ponens, direct proof, proof by contradiction, contrapositive, logically equivalent, logically equivalence, De Morgans laws, contradiction, set, member, element, equality of sets, natural numbers $\mathbb{N}$*, integers* $\mathbb{Z}$*, rational numbers* $\mathbb{Q}$*, real numbers* $\mathbb{R}$*, irrational numbers (i.e., real numbers that are not rational), even integer, odd integer, empty set, variable, existential quantifier, universal quantifier, universal set, subset, proper subset*

Problems. These problems are intended to prepare you for the midterm, though not all of them are problems I would put on a midterm. Also consider the exercises in the book (both assigned and unassigned).

1. Show that the following statement forms are logically equivalent.

$$S_1 : (P \vee Q) \Rightarrow R.$$

$$S_2 : \big((\sim P) \wedge (\sim Q)\big) \vee R.$$

2. Let $x, y \in \mathbb{Z}$. Prove that $(x+1)(2x+y)$ is odd if and only if x is even and y is odd.

3. Let $x, y, z, \in \mathbb{R}$. Prove that if $6x + 6y + 4 < 13z$, then $2x < 3z$ or $3y + 2 < 2z$.

4. Answer the following questions. Express your answer in a complete sentence. Change quantifiers as appropriate.
   (a) State the negation of the following statement:
   There is a real number z *so that* $z - z^2 > 1$.
   (b) State the negation of the following statement:
   For each $n \in \mathbb{N}$*, if* $n! > 4^n$*, then* $n! + 1$ *is a prime number.*
   (c) Let $m \in \mathbb{Z}$. State the contrapositive of the following implication:
   If m^2 *is even, then* $m^3 - 1$ *is odd.*

(d) Let $x \in \mathbb{R}$. State the contrapositive of the following implication:
If $x(x-2)=0$, then $x=0$ or $x=2$.

5. For statements P and Q, show that $\big((\sim Q) \wedge (P \Rightarrow Q)\big) \Rightarrow (\sim P)$ is a tautology.

6. Prove the following statement is true:
For all integers a and b, if a is odd and b is even, then $\frac{ab+3b}{4}$ is an integer.

7. Prove the following statement is true:
For all $n \in \mathbb{Z}$, if $n(n+3)$ is odd, then $n^3 - 2n \geq 0$.

8. Prove that if x is a real number so that $1 < x \leq 3$, then $\frac{8}{x^2-1} \geq 1$.
(*Hint:* Observe that $x^2 - 1 = (x+1)(x-1)$.)

9. Consider the following statement:

P : For all non-zero rational numbers x and all irrational numbers y, the product xy is irrational.

(a) Write the negation of P in a sentence. (You must appropriately alter the quantifiers.)

(b) Prove the statement P is true. (*Hint:* Try a proof by contradiction.)

10. In each of the following parts, a universal set is specified, and then a quantified statement is expressed. Write negations of these quantified statements.

(a) Let $S \subset \mathbb{R}$. For every $x \in S$, $x^4 \geq x^2$.

(b) Let $S \subset \mathbb{R}$. There exist elements $s \in S$ and $t \in S$ so that $st = 1$.

(c) Let A be a set of non-negative real numbers. For any $a \in A$, if $\sqrt{a}$ is a rational number, then $\sqrt{a}$ is an integer.

(d) Let $X \subset \mathbb{R}$ and $m \in \mathbb{R}$. We have $-m \leq x \leq m$ for any $x \in X$.

(e) Let $X \subset \mathbb{R}$. For every non-empty subset Y of X, there is an $m \in Y$ with the property that $m \leq y$ for any $y \in Y$.

11. Consider the implication

$$P: \text{ If } x^2 = 4, \text{ then } x^2 = x + 6.$$

For which $x \in \mathbb{R}$ is P true? For which $x \in \mathbb{R}$ is P false?

12. Describe each of the following sets in the format $\{x \in \mathbb{N} : P(x)\}$.

(a) $\{3, 6, 9, 12, 15, \ldots\}$.

(b) $\{1, 10, 100, 1000, 10000, \ldots\}$.

(c) $\{1, 5, 437\}$.

13. Express the following statements using quantifiers. Include any implicit universal quantifiers.

(a) Every positive integer is the sum of two squares of integers.

(b) If x is a positive real number, then so is $\arctan x$.

(c) There is a smallest natural number.

(d) If $b^2 - 4ac \geq 0$, then $ax^2 + bx + c$ has a real root.

14. (a) Complete the following definition.
If A and B are sets, then $A \subseteq B$ if ...

(b) Complete the following definition.
If A and B are sets, then $A = B$ if ...

(c) Let $\emptyset$ be the empty set and A be another set. Prove that $\emptyset \subseteq A$.

(d) Let A, B and C be sets. Prove that if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

(e) Prove that if A is a set and $A \subseteq \emptyset$, then $A = \emptyset$.