Left and right inverses
and other definitions involving functions

Definition. If $f : A \to B$ and $X \subseteq A$, then the image of X is

$$f(X) = \{f(x) : \ x \in X\}.$$

Definition. Given sets A and B, a function $f : A \to B$ is said to be *surjective* or *onto* B if $f(A) = B$, and f is a *surjection*. Equivalently, for each $b \in B$ there exists an element $a \in A$ such that $f(a) = b$.

Definition. A function $f : A \to B$ is said to be *one-to-one* (abbreviated $1-1$), *injective*, or an *injection* if the following implication is true for every $a_1, a_2 \in A$:

$$f(a_1) = f(a_2) \implies a_1 = a_2.$$

Definition. If A is a set, the *identity map* on A is the map $id_A : A \to A$ defined so that $id_A(a) = a$ for all $a \in A$.

Definition. A *left inverse* of a function $f : A \to B$ is a function $\ell : B \to A$ so that $\ell \circ f = id_A$.

Definition. A *right inverse* of a function $f : A \to B$ is a function $r : B \to A$ so that $f \circ r = id_B$.

Result 1. Assume $A \neq \emptyset$. Then a function $f : A \to B$ has a left inverse if and only if f is injective.

Result 2. A function $f : A \to B$ has a right inverse if and only if f is surjective.

Result 3. If $f : A \to B$ has both a left inverse $\ell : B \to A$ and a right inverse $r : B \to A$, then $r = \ell$.

Definition. An *inverse* of a function $f : A \to B$ is a function $g : B \to A$ so that $f \circ g = id_B$ and $g \circ f = id_A$ (so g is both a right and a left inverse). If f has an inverse, it is unique. We usually denote the inverse of f using f^{-1}.

Definition. A function $f : A \to B$ is said to be *bijective*, a *bijection*, or a *one-to-one correspondence* if it is both injective and surjective.

Result 4. Let $f : A \to B$ be a function. Then, the following are equivalent:

- f is a bijection (i.e., $1-1$ and onto).
- f has both a left and a right inverse.
- f has an inverse.

(That is, the three statements above are either all true or all false.)

Proof of Result 1. First assume that $f : A \to B$ has a left inverse $\ell : B \to A$. We will show that f is injective. That is, we will show:

$$\forall a_1, a_2 \in A, \quad f(a_1) = f(a_2) \implies a_1 = a_2.$$

Let $a_1, a_2 \in A$ and assume $f(a_1) = f(a_2)$. Since $f(a_1) = f(a_2)$ we know $\ell\big(f(a_1)\big) = \ell\big(f(a_2)\big)$. Then since ℓ is a left inverse,

$$\ell\big(f(a_1)\big) = \ell \circ f(a_1) = id_A(a_1) = a_1 \quad \text{and} \quad \ell\big(f(a_2)\big) = \ell \circ f(a_2) = id_A(a_2) = a_2,$$

so $a_1 = a_2$.

Now suppose that f is injective. We will construct an left inverse $g : B \to A$. Since $A \neq \emptyset$, we can pick an $a_0 \in A$. For $b \in B$ define $g(b) = a$ if $f(a) = b$, and define $g(b) = a_0$ if there is no a so that $f(a) = b$. To see g is well defined, we need to know that there is at most one a so that $f(a) = b$, but this is true because f is $1 - 1$. To see that g is a left inverse, pick any $a \in A$. We'll show $g \circ f(a) = a$ so that $g \circ f = id_A$. Let $b = f(a)$. Then by definition of g, we have $g(b) = a$. Therefore $g \circ f(a) = g(b) = a$ as desired. □

Proof of Result 2. First assume that $f : A \to B$ has a right inverse $r : B \to A$. We will show that f is surjective. That is, we will show:

$$\forall b \in B,\ \exists a \in A \text{so that} f(a) = b.$$

Let $b \in B$. Then set $a = r(b)$. Since r is a right inverse, $f \circ r(b) = id_B(b) = b$. On the other hand, $f \circ r(b) = f\big(r(b)\big) = f(a)$. So, $f(a) = b$ as desired.

Now suppose that f is surjective. We will construct a right inverse $g : B \to A$. Since f is surjective, for each $b \in B$, the preimage of b,

$$P_b = \{a \in A \ : \ f(a) = b\}$$

is non-empty. So for each $b \in B$ we can pick a $g(b) \in P_b$. This defines a function $g : B \to A$. To see it is a right inverse, we must show that $f \circ g = id_B$. Pick any $b \in B$. Then $g(b) \in P_b$ so $f \circ g(b) = b$.
□