Math A4500: Midterm 3 Study Guide

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Disclaimer. This test is just a recommendation of things to study. You may be asked about things that do not appear here.

Sections that will be covered. §2.1-2.5. An understanding of earlier material may be necessary to do the problems on the midterm, but earlier material will not be explicitly tested.

Definitions. You will be asked to define several terms on the test. The following is a list of terms which might appear.

diffeomorphism (of $\mathbb{R}^2$*), stable and unstable sets of a point, forward and backward asymptotic, homoclinic, heteroclinic, Markov partition, trapping region, attractor, inverse limit space*

You are expected to know the definitions given in the book.

Theorems. Theorems and results from the book you should be able to prove.

- Theorem 1.21: The contraction mapping theorem.
- Proposition 5.2: Attractors are invariant.

Techniques. You should be able to use the various techniques we have developed for understanding particular dynamical systems acting on subsets of the real line. For instance, you should be able to:

- Identify the stable and unstable sets of points in simple systems (such as linear maps of $\mathbb{R}^2$ and $\mathbb{R}^3$, horseshoe like systems, toral automorphisms.)
- Coding via a 2-sided shift space as with the horseshoe.
- Coding via a Markov partition (for toral automorphisms and solenoids for instance).

Types of problems. There will be between four and six multi-part problems on the midterm. You'll have the full class period (100 minutes) to complete the midterm. Problems of the following forms are likely to appear:

- *Homework*: Problems similar to assigned homework problems.
- *Recall Something, then prove something.*: Problems that as you to recall a definition or a major result, and then use it in a proof.
- *Math comprehension:* The problem states a new definition, which may not have seen before, and asks you to use the definition in basic proofs.
- *Prove a result:* Problems which ask you to prove a result which is proved in the book. See the section titled "Theorems" above.

1. Complete the following definitions:

   (a) A function $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a *diffeomorphism* if ...

   Solution: F is one-to-one, onto and C^∞ and its inverse is C^∞.

   (b) Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be continuous. A closed region $N \subset \mathbb{R}^n$ is a *trapping region* for F if ...

   Solution: $F(N) \subset int(N)$.

2. Consider the matrix $M = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{2} \end{bmatrix}$.

   (a) Is the matrix hyperbolic? Why or why not?

   Solution: We will compute the eigenvalues. The characteristic polynomial is

$$det(M - \lambda I) = (\frac{1}{2} - \lambda)^2 + \frac{1}{16} = \lambda^2 - \lambda + \frac{5}{16}.$$

   By the quadratic formula:

$$\lambda = \frac{1 \pm \sqrt{1 - (5/4)}}{2} = \frac{1}{4}(2 \pm i).$$

   The absolute value of the eigenvalues is given by:

$$|\lambda| = \frac{1}{4}\sqrt{2^2 + 1} = \frac{\sqrt{5}}{4} < 1.$$

   So the matrix is hyperbolic.

   (b) Consider the action of multiplication by M on the plane:

$$M(x, y) = \big(\frac{2x + y}{4}, \frac{-x + 2y}{4}\big).$$

   Describe the forward orbit of points under M. For each $(x, y) \in \mathbb{R}^2$, what happens to $M^n(x, y)$ as $n \to \infty$.

   Solution: The matrix M can be written as $M = \frac{\sqrt{5}}{4}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, where θ satisfies $\cos\theta = \frac{2}{\sqrt{5}}$ and $\sin\theta = \frac{-1}{\sqrt{5}}$. So, M is a rotation composed with a contraction by $\frac{\sqrt{5}}{4}$. Therefore,

$$|M\mathbf{v}| = \frac{\sqrt{5}}{4}|\mathbf{v}| \quad \text{for all } \mathbf{v} \in \mathbb{R}^2.$$

Therefore,

$$|M^n\mathbf{v}| = (\frac{\sqrt{5}}{4})^n|\mathbf{v}| \quad \text{for all } \mathbf{v} \in \mathbb{R}^2.$$

Thus $|M^n\mathbf{v}| \to 0$ as $n \to \infty$.

(c) Recall that $M(x, y)$ has *sensitive dependence on initial conditions* if there exists a $\delta > 0$ such that for any $(x, y) \in \mathbb{R}^2$ and any neighborhood N of (x, y), there is a point $(x', y') \in N$ and an $n \geq 0$ so that $|M^n(x, y) - M^n(x', y')| > \delta$. Prove that $M(x, y)$ does **not** have sensitive dependence on initial conditions.

Solution: Assume to the contrary that $M(x, y)$ did have sensitive dependence on initial conditions. Then, there would be a $\delta > 0$ so that for any $(x, y) \in \mathbb{R}^2$ and any neighborhood N of (x, y), there is a point $(x', y') \in N$ and an $n \geq 0$ so that $|M^n(x, y) - M^n(x', y')| > \delta$. We will derive a contradiction. We are given δ. Choose $(x, y) = (0, 0)$ and choose

$$N = \{\mathbf{v} \in \mathbb{R}^2 \ : \ |\mathbf{v}| < \delta\}.$$

Then, for any $\mathbf{v} \in N$ and any $n \geq 0$, we have

$$|M^n(0, 0) - M^n\mathbf{v}| = (\frac{\sqrt{5}}{4})^n|\mathbf{v}| \leq |\mathbf{v}| < \delta.$$

But, this is a contradiction.

3. Consider the matrix $M = \begin{bmatrix} 4 & 1 \\ -1 & 0 \end{bmatrix}$. The matrix has integer entries and determinent one and so induces a homeomorphism L of the square torus $T = \mathbb{R}^2/\mathbb{Z}^2$.

(a) Let $[x, y] \in T$ be an arbitrary point. Explicitly describe the stable and unstable sets of the point.

Solution: We will compute the eigenvalues. The characteristic polynomial is

$$det(M - \lambda I) = (4 - \lambda)(-\lambda) + 1 = \lambda^2 - 4\lambda + 1.$$

By the quadratic formula:

$$\lambda = \frac{4 \pm \sqrt{16 - 4}}{2} = 2 \pm \sqrt{3}.$$

So, $0 < \lambda_s = 2 - \sqrt{3} < 1$ and $\lambda_u = 2 + \sqrt{3} > 1$. We conclude that the matrix is hyperbolic.

We can compute the stable and unstable eigenvectors from the equations

$$(M - \lambda_s)\mathbf{v}_s = \mathbf{0} \quad \text{and} \quad (M - \lambda_u)\mathbf{v}_u = \mathbf{0}.$$

We can take

$$\mathbf{v}_s = (2-\sqrt{3}, -1) \quad \text{and} \quad \mathbf{v}_u = (2+\sqrt{3}, -1).$$

The stable and unstable sets of the point $[x, y] \in T$ are

$$W^s[x,y] = \{[x + t(2-\sqrt{3}), y - t] \ : \ t \in \mathbb{R}\} \quad \text{and}$$
$$W^u[x,y] = \{[x + t(2+\sqrt{3}), y - t] \ : \ t \in \mathbb{R}\}.$$

(b) Explain how to construct a point in the torus which is backward asymptotic to $[0,0] \in T$ and which is forward asymptotic to the fixed point $[\frac{1}{2}, \frac{1}{2}] \in T$. (A brief description and a labeled diagram will suffice.)

Solution: We may choose any point on $W^u[0,0] \cap W^s[\frac{1}{2}, \frac{1}{2}]$. From the prior part, we have

$$W^u[0,0] = \{[t(2+\sqrt{3}), -t] \ : \ t \in \mathbb{R}\} \quad \text{and}$$
$$W^s[\tfrac{1}{2}, \tfrac{1}{2}] = \{[\tfrac{1}{2} + s(2-\sqrt{3}), \tfrac{1}{2} - s] \ : \ s \in \mathbb{R}\}.$$

To find a common point we can solve the equations:

$$t(2+\sqrt{3}) = \tfrac{1}{2} + s(2-\sqrt{3}) \quad \text{and} \quad -t = \tfrac{1}{2} - s.$$

The second equation says $t = \frac{1}{2} - s$. Then, we substitute into the first equation:

$$(\frac{1}{2} - s)(2+\sqrt{3}) = \tfrac{1}{2} + s(2-\sqrt{3}).$$

Solving yields $s = \frac{1}{8}(1+\sqrt{3})$. Therefore, such a point is given by

$$[\tfrac{1}{2} + s(2-\sqrt{3}), \tfrac{1}{2} - s] = [\frac{1}{8}(3+\sqrt{3}), \frac{1}{8}(3-\sqrt{3})].$$

(c) What term is used to describe the point found in part (b)?

Solution: The point is heteroclinic to $[0,0]$ and $[\frac{1}{2}, \frac{1}{2}]$.

4. Consider the full 2-sided shift space Σ_2 on the alphabet $\{0,1\}$. This is the space of all sequences

$$\mathbf{s} = (\dots s_{-1} \cdot s_0 s_1 \dots) \quad \text{with } s_i \in \{0,1\} \text{ for all } i \in \mathbb{Z}.$$

The shift map is the map

$$\sigma : \Sigma_2 \to \Sigma_2; \quad (\dots s_{-1} \cdot s_0 s_1 \dots) \mapsto (\dots s_{-1} s_0 \cdot s_1 s_2 \dots).$$

The space Σ_2 becomes a metric space when given the metric

$$d(\mathbf{s}, \mathbf{t}) = \sum_{i=-\infty}^{\infty} \frac{|s_i - t_i|}{2^{|i|}}.$$

(a) Let $\mathbf{0} = (\ldots 0 \cdot 00 \ldots)$. Prove that there is a dense set of points $\mathbf{s} \in \Sigma_2$ which are homoclinic to $\mathbf{0}$.

Solution: Let H denote the set of points homoclinic to $\mathbf{0}$. We will prove that $H \cup \{\mathbf{0}\}$ is dense in T. Note that this directly implies that H is dense in T since $\mathbf{0}$ is just a single point. (We mention this because technically $\mathbf{0}$ is not be homoclinic to $\mathbf{0}$.)

Fix any point $\mathbf{t} = (\ldots t_{-1} \cdot t_0 t_1 \ldots) \in \Sigma_2$ and choose any $\epsilon > 0$. We may assume that $\mathbf{t} \neq \mathbf{0}$ since $\Sigma_2 \setminus \{\mathbf{0}\}$ is dense in Σ_2. We will find a point $\mathbf{s} \in H$ within ϵ of $\mathbf{t}$. Choose a positive integer n so that $\frac{1}{2^{n-1}} < \epsilon$. Then define $\mathbf{s} = (\ldots s_{-1} \cdot s_0 s_1 \ldots)$ so that

$$s_k = \begin{cases} t_k & \text{if } |k| \leq n \\ 0 & \text{otherwise.} \end{cases}$$

This point is both forward and backward asymptotic to $\mathbf{0}$ since it begins and ends with all zeros. Thus $\mathbf{s} \in H \cup \{\mathbf{0}\}$. Also,

$$d(\mathbf{s}, \mathbf{t}) = \sum_{i=-\infty}^{\infty} \frac{|s_i - t_i|}{2^{|i|}} \leq 2 \sum_{i=n+1}^{\infty} \frac{1}{2^i} = 2(\frac{1}{2^n}) = \frac{1}{2^{n-1}} < \epsilon.$$

(b) Let $\mathbf{1} = (\ldots 1 \cdot 11 \ldots)$. It is also true that there is a dense set points which are homoclinic to $\mathbf{1}$. Use this fact and part (a) to prove that $\sigma : \Sigma_2 \to \Sigma_2$ has sensitive dependence on initial conditions. (See the definition in problem 3c.)

Solution: Set $\delta = \frac{1}{4} d(\mathbf{0}, \mathbf{1})$. Choose any point $[x, y] \in T$ and a neighborhood N of $[x, y]$. By assumption, we can find a point $[x_0, y_0] \in N$ homoclinic to $\mathbf{0}$ and a point $[x_1, y_1] \in N$ homoclinic to $\mathbf{1}$. Then, by definition of homoclinic, we can choose $n > 0$ so that

$$d\big(\sigma^n[x_0, y_0], \mathbf{0}\big) < \delta \quad \text{and} \quad d\big(\sigma^n[x_1, y_1], \mathbf{1}\big) < \delta.$$

By the triangle inequality, we have

$$d\big(\mathbf{0}, \sigma^n[x_0, y_0]\big) + d\big(\sigma^n[x_0, y_0], \sigma^n[x, y]\big) + d\big(\sigma^n[x, y], \sigma^n[x_1, y_1]\big) + d\big(\sigma^n[x_1, y_1], \mathbf{1}\big) \geq d(\mathbf{0}, \mathbf{1}).$$

Combining the inequalities yields:

$$d\big(\sigma^n[x_0, y_0], \sigma^n[x, y]\big) + d\big(\sigma^n[x, y], \sigma^n[x_1, y_1]\big) > d(\mathbf{0}, \mathbf{1}) - 2\delta = 2\delta.$$

So, either $d\big(\sigma^n[x_0, y_0], \sigma^n[x, y]\big) > \delta$ or $d\big(\sigma^n[x, y], \sigma^n[x_1, y_1]\big) > \delta$. This proves that σ has sensitive dependence of initial conditions with constant $\delta = \frac{1}{4}d(\mathbf{0}, \mathbf{1})$. (Actually any $\delta < \frac{1}{2}d(\mathbf{0}, \mathbf{1})$ works.)

5. (a) Let $F : X \to X$ be a continuous map with $X \subset \mathbb{R}^n$. What is a *trapping region for* F? (Give a formal definition.)

Solution: A closed region $N \subset \mathbb{R}^n$ is a *trapping region* for F if $F(N)$ is contained in the interior of N. (Note: Devaney actually seems to assume N is compact.)

(b) What is an *attractor* for F? (Give a formal definition.)

Solution: A set Λ is called an attractor for F if there is a neighborhood N of Λ (i.e. an open set containing Λ) for which the closure of N is is a trapping region and $\Lambda = \bigcap_{n \geq 0} F^n(N)$.

(c) Consider the linear map $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by $F(x, y) = (\frac{y}{4}, 2x)$. Find a bounded trapping region for the map and describe the associated attractor.

Solution: Let

$$N = [-1, 1] \times [-3, 3] = \{(x, y) : \ -1 \leq x \leq 1 \text{ and } -3 \leq y \leq 3\}.$$

Observe that if $(x, y) \in N$ and $(x', y') = F(x, y) = (\frac{y}{4}, 2x)$ then

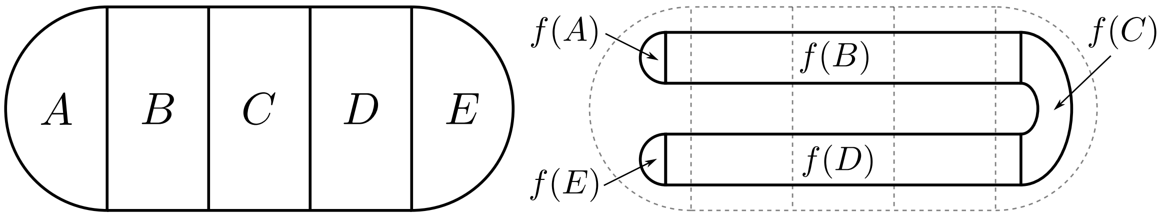
$$\frac{-1}{4} \leq x' \leq \frac{1}{4} \quad \text{and} \quad -2 \leq y' \leq 2.$$

Thus $F(N) = [\frac{-1}{4}, \frac{1}{4}] \times [-2, 2]$ is in the interior of N so N is a trapping region. Observe that

$$F^2(x, y) = F(\frac{y}{4}, 2x) = (\frac{x}{2}, \frac{y}{2}).$$

Thus F^2 is a linear contraction. It follows that all orbits tend to the origin. Thus the attractor must be $\{(0, 0)\}$.

6. (The Hoseshoe Map) Consider the region $X \subset \mathbb{R}^2$ which is a union of the five closed subregions A, B, C, D, and E. Here A and E are half-disks with diameter two and B, C and D are 1×2 rectangles. We define the continuous injective map $f : X \to X$ as indicated by the diagram below:



The matrix of partial derivatives Df is constant on each of the interiors of the regions A, B, D and E and is given by:

$$Df|_{int(A)} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}. \qquad Df|_{int(B)} = \begin{bmatrix} \frac{7}{2} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}.$$

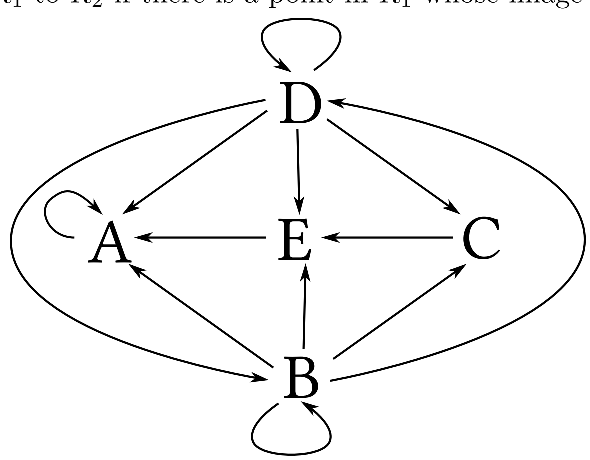
$$Df|_{int(D)} = \begin{bmatrix} \frac{-7}{2} & 0 \\ 0 & \frac{-1}{4} \end{bmatrix}. \qquad Df|_{int(E)} = \begin{bmatrix} \frac{-1}{4} & 0 \\ 0 & \frac{-1}{4} \end{bmatrix}.$$

Questions:

(a) Explain why A contains a unique fixed point, $p \in A$.

> **Solution:** The restriction $f|_A$ is a contraction (contracting distances by a factor of $\frac{1}{4}$). By the contraction mapping theorem, f has a unique fixed point.

(b) Let $\Lambda_+ \subset X$ denote the set of points which are not forward asymptotic to p. Describe the possible sequences $\{s_n : n \geq 0\}$ with each $s_n \in \{A, B, C, D, E\}$ so that there is a $q \in \Lambda_+$ so that $f^n(q)$ lies in the region s_n for all $n \geq 0$.

> **Solution:** Observe the following graph of possible transitions (i.e., an arrow is drawn from R_1 to R_2 if there is a point in R_1 whose image is in R_2).
>
> 
>
>
> Since $f|_A$ is a contraction, no sequence for $q \in \Lambda_+$ can include an A. Since $f(E) \subset A$ and $f(C) \subset E$, no such sequence can include C or E. Thus, the possible sequences are given by those sequences which only include B and D.

(c) Prove that the region D contains a fixed point.

Solution: Observe that by the Fundamental Theorem of Calculus we have

$$f|_D(x, y) = (\tfrac{-7}{2}x + c, \tfrac{-1}{4}y + d)$$

for some constants c and d. The map on the y-coordinate $y \mapsto \frac{-1}{4}y + d$ is a contraction and so has a unique fixed point y_0 (and because the interval of y-coordinates in D is mapped into itself by the map this y_0 is a y-coordinate for points in D). For the x-coordinate we can use the same argument: the map $f^{-1}|_D$ is a contraction in the x-coordinate. So there is a x_0 which is an x-coordinate for points in D which is fixed. Then the point (x_0, y_0) is fixed by f.

(d) Describe the local stable and unstable manifolds of the point found in part (c). (Do not attempt to describe the global stable and unstable manifolds.)

Solution: The stable manifold is locally vertical. The unstable manifold is locally horizontal.

7. (The Solenoid) Let $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ be the circle with coordinates measured modulo 2π. Let $B = \{(x, y) \ : \ x^2 + y^2 \leq 1\}$ be the unit disk in the plane, and define $D = S^1 \times B$. Consider the map $F : D \to D$ defined by

$$F(\theta, p) = (2\theta, \frac{1}{10}p + \frac{1}{2}e^{i\theta}),$$

where $e^{i\theta} = (\cos\theta, \sin\theta)$. The map F is injective and $F(D) \subset$ *int* D. The *solenoid* is

$$\Lambda = \bigcap_{j=0}^{\infty} F^j(D).$$

Let Σ be the inverse limit space for the doubling map $g(\theta) = 2\theta$. That is,

$$\Sigma = \{\theta = (\theta_0\theta_1\theta_2 \ldots) \ : \ \theta_j \in S^1 \text{ and } g(\theta_{j+1}) = \theta_j \text{ for all integers } j \geq 0\}.$$

(a) Define three maps satisfying the following statements:

(1) The *shift map* homeomorphism, $\sigma : \Sigma \to \Sigma$.

(2) The inverse of the shift map, $\sigma^{-1} : \Sigma \to \Sigma$.

(3) A homeomorphism $S : \Lambda \to \Sigma$ so that $S \circ F = \sigma \circ S$.

(You just need to define the maps, you do not need to prove the statements are satisfied.)

Solution: The shift map is

$$\sigma(\theta_0\theta_1\theta_2 \ldots) = \big(g(\theta_0)\theta_0\theta_1\theta_2 \ldots\big).$$

The inverse of the shift map is given by

$$\sigma^{-1}(\theta_0\theta_1\theta_2\ldots) = \big(\theta_1\theta_2\ldots\big). \tag{1}$$

Observe that on Λ, F^{-n} is defined for all integers $n \geq 0$. For $(\theta, p) \in \Lambda$ and $n \geq 0$ define $(\theta_n, p_n) = F^{-n}(\theta, p) \in \Lambda$. The map S is defined

$$S(\theta, p) = (\theta_0\theta_1\theta_2\ldots) \in \Sigma.$$

(b) Prove that the unstable set, $W^u(\mathbf{0})$, for the map σ of the point $\mathbf{0} = (000\ldots)$ is given by

$$\Big\{\big([x], [\tfrac{x}{2}], [\tfrac{x}{4}], [\tfrac{x}{8}], \ldots\big) \in \Sigma \;:\; x \in \mathbb{R}\Big\}.$$

Here, we use $[x]$ to denote the equivalence class in $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ of the real number x.

Solution: First we need to show that if $\Theta = \big([x], [\frac{x}{2}], [\frac{x}{4}], [\frac{x}{8}], \ldots\big) \in \Sigma$ for some $x \in \mathbb{R}$ then $\Theta \in W^u(\mathbf{0})$. That is, we need to prove that

$$\lim_{n\to\infty} d\big(\sigma^{-n}(\Theta), \mathbf{0}\big) = 0.$$

Observe from (1) that

$$\sigma^{-n}(\Theta) = \Big([\frac{x}{2^n}], [\frac{x}{2^{n+1}}], [\frac{x}{2^{n+2}}], \ldots\Big).$$

This sequence converges to $\mathbf{0}$ because for any $k \geq 0$ the k-th position of $\sigma^{-n}(\Theta)$ is $[\frac{x}{2^{n+k}}]$ which tends to $[0]$ as $n \to \infty$. This this holds for all k, $\lim_{n\to\infty} d\big(\sigma^{-n}(\Theta), \mathbf{0}\big) = 0$. (One can also use the metric for this.)

8. (Toral Automorphisms) Let T denote the standard torus, $T = \mathbb{R}^2/\mathbb{Z}^2$. Recall that if $x, y \in \mathbb{R}$, we use $[x, y]$ to represent the corresponding point in the torus, which is an equivalence class,

$$[x, y] = \{(x + m, y + n) \;:\; m, n \in \mathbb{Z}\}.$$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2×2 matrix with integer entries and determinant ± 1. Suppose also that the matrix is hyperbolic. This guarantees that there are two real eigenvalues λ_s and λ_u with $0 < |\lambda_s| < 1$ and $|\lambda_u| > 1$. Then multiplication my A induces a hyperbolic automorphism of the torus $L : T \to T$.

(a) Prove that every point with rational coordinates is periodic.

Solution: Let $[x, y]$ be a point in the torus with rational coordinates. Then there are $p, q, n \in \mathbb{Z}$ with $n \neq 0$ so that

$$x = \frac{p}{n} \quad \text{and} \quad y = \frac{q}{n}.$$

Observe that

$$L([x, y]) = [\frac{ap + bq}{n}, \frac{cp + dq}{n}]. \tag{2}$$

Now observe that there are no more than n^2 points in the set

$$R_n = \{[\frac{i}{n}, \frac{j}{n}] \ : \ i, j, n \in \mathbb{Z}\}.$$

The equation (2) tells us that R_n is invariant under L. Thus, points can have an orbit of size at most n^2. Since L is invertible, the orbit must be periodic (as opposed to preperiodic).

(b) Let $p \in T$. Describe the stable and unstable sets of p.

Solution: Let $v_u \in \mathbb{R}^2$ be an expanding eigenvector for A (with eigenvalue λ_u), and let $v_s \in \mathbb{R}^2$ be a contracting eigenvector (with eigenvalue λ_s). The stable set of L is

$$W^s([p]) = \{[p + tv_s] \ : \ t \in \mathbb{R}\}$$

and the unstable set is

$$W^u([p]) = \{[p + tv_u] \ : \ t \in \mathbb{R}\}.$$

(c) Is it true that there are a dense set of points which are homoclinic to $[0, 0]$? Explain why or why not.

Solution: Yes. The sets $W^s([0, 0])$ and $W^u([0, 0])$ are dense and everywhere transverse, so they have a dense collection of intersections.

(d) Sketch a proof that $L : T \to T$ is topologically transitive. (Just give the main ideas in such a proof.)

Solution: Recall by definition L is topologically transitive if for any two open sets $U, V \subset T$ there is a point $p \in U$ and an $n \geq 1$ so that $L^n(p) \in V$. Fix U and V. By density of periodic points there is a periodic point $x \in V$. Let k be the period of x. Then x has a dense set of homoclinic points, so we can find a point $p \in U$ which is homoclinic to x. Then

$$\lim_{n \to \infty} d\big(L^{nk}(p), x\big) = 0,$$

so for n sufficiently large, $L^{nk}(p) \in V$.