Math A4500: Midterm 3 Study Guide

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Disclaimer. This test is just a recommendation of things to study. You may be asked about things that do not appear here.

Sections that will be covered. §2.1-2.5. An understanding of earlier material may be necessary to do the problems on the midterm, but earlier material will not be explicitly tested.

Definitions. You will be asked to define several terms on the test. The following is a list of terms which might appear.

diffeomorphism (of $\mathbb{R}^2$*), stable and unstable sets of a point, forward and backward asymptotic, homoclinic, heteroclinic, Markov partition, trapping region, attractor, inverse limit space*

You are expected to know the definitions given in the book.

Theorems. Theorems and results from the book you should be able to prove.

- Theorem 1.21: The contraction mapping theorem.
- Proposition 5.2: Attractors are invariant.

Techniques. You should be able to use the various techniques we have developed for understanding particular dynamical systems acting on subsets of the real line. For instance, you should be able to:

- Identify the stable and unstable sets of points in simple systems (such as linear maps of $\mathbb{R}^2$ and $\mathbb{R}^3$, horseshoe like systems, toral automorphisms.)
- Coding via a 2-sided shift space as with the horseshoe.
- Coding via a Markov partition (for toral automorphisms and solenoids for instance).

Types of problems. There will be between four and six multi-part problems on the midterm. You'll have the full class period (100 minutes) to complete the midterm. Problems of the following forms are likely to appear:

- *Homework*: Problems similar to assigned homework problems.
- *Recall Something, then prove something.*: Problems that as you to recall a definition or a major result, and then use it in a proof.
- *Math comprehension:* The problem states a new definition, which may not have seen before, and asks you to use the definition in basic proofs.
- *Prove a result:* Problems which ask you to prove a result which is proved in the book. See the section titled "Theorems" above.

1. Complete the following definitions:
   (a) A function $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a *diffeomorphism* if ...
   (b) Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be continuous. A closed region $N \subset \mathbb{R}^n$ is a *trapping region* for F if ...

2. Consider the matrix $M = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{2} \end{bmatrix}$.
   (a) Is the matrix hyperbolic? Why or why not?
   (b) Consider the action of multiplication by M on the plane:

   $$M(x, y) = (\frac{2x + y}{4}, \frac{-x + 2y}{4}).$$

   Describe the forward orbit of points under M. For each $(x, y) \in \mathbb{R}^2$, what happens to $M^n(x, y)$ as $n \to \infty$.
   (c) Recall that $M(x, y)$ has *sensitive dependence on initial conditions* if there exists a $\delta > 0$ such that for any $(x, y) \in \mathbb{R}^2$ and any neighborhood N of (x, y), there is a point $(x', y') \in N$ and an $n \geq 0$ so that $|M^n(x, y) - M^n(x', y')| > \delta$. Prove that $M(x, y)$ does **not** have sensitive dependence on initial conditions.

3. Consider the matrix $M = \begin{bmatrix} 4 & 1 \\ -1 & 0 \end{bmatrix}$. The matrix has integer entries and determinent one and so induces a homeomorphism L of the square torus $T = \mathbb{R}^2/\mathbb{Z}^2$.
   (a) Let $[x, y] \in T$ be an arbitrary point. Explicitly describe the stable and unstable sets of the point.
   (b) Explain how to construct a point in the torus which is backward asymptotic to $[0, 0] \in T$ and which is forward asymptotic to the fixed point $[\frac{1}{2}, \frac{1}{2}] \in T$. (A brief description and a labeled diagram will suffice.)
   (c) What term is used to describe the point found in part (b)?

4. Consider the full 2-sided shift space Σ_2 on the alphabet $\{0, 1\}$. This is the space of all sequences

   $$\mathbf{s} = (\dots s_{-1} \cdot s_0 s_1 \dots) \quad \text{with } s_i \in \{0, 1\} \text{ for all } i \in \mathbb{Z}.$$

   The shift map is the map

   $$\sigma : \Sigma_2 \to \Sigma_2; \quad (\dots s_{-1} \cdot s_0 s_1 \dots) \mapsto (\dots s_{-1} s_0 \cdot s_1 s_2 \dots).$$

   The space Σ_2 becomes a metric space when given the metric

   $$d(\mathbf{s}, \mathbf{t}) = \sum_{i=-\infty}^{\infty} \frac{|s_i - t_i|}{2^{|i|}}.$$

   (a) Let $\mathbf{0} = (\dots 0 \cdot 00 \dots)$. Prove that there is a dense set of points $\mathbf{s} \in \Sigma_2$ which are homoclinic to $\mathbf{0}$.

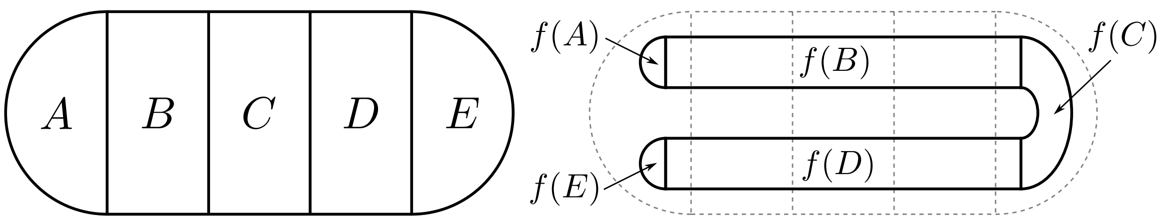
(b) Let $\mathbf{1} = (\ldots 1 \cdot 11 \ldots)$. It is also true that there is a dense set points which are homoclinic to $\mathbf{1}$. Use this fact and part (a) to prove that $\sigma : \Sigma_2 \to \Sigma_2$ has sensitive dependence on initial conditions. (See the definition in problem 3c.)

5. (a) Let $F : X \to X$ be a continuous map with $X \subset \mathbb{R}^n$. What is a *trapping region for* F? (Give a formal definition.)

   (b) What is an *attractor* for F? (Give a formal definition.)

   (c) Consider the linear map $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by $F(x, y) = (\frac{y}{4}, 2x)$. Find a bounded trapping region for the map and describe the associated attractor.

6. (The Hoseshoe Map) Consider the region $X \subset \mathbb{R}^2$ which is a union of the five closed subregions A, B, C, D, and E. Here A and E are half-disks with diameter two and B, C and D are 1×2 rectangles. We define the continuous injective map $f : X \to X$ as indicated by the diagram below:



The matrix of partial derivatives Df is constant on each of the interiors of the regions A, B, D and E and is given by:

$$Df|_{int(A)} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}. \qquad Df|_{int(B)} = \begin{bmatrix} \frac{7}{2} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}.$$

$$Df|_{int(D)} = \begin{bmatrix} \frac{-7}{2} & 0 \\ 0 & \frac{-1}{4} \end{bmatrix}. \qquad Df|_{int(E)} = \begin{bmatrix} \frac{-1}{4} & 0 \\ 0 & \frac{-1}{4} \end{bmatrix}.$$

Questions:

(a) Explain why A contains a unique fixed point, $p \in A$.

(b) Let $\Lambda_+ \subset X$ denote the set of points which are not forward asymptotic to p. Describe the possible sequences $\{s_n : n \geq 0\}$ with each $s_n \in \{A, B, C, D, E\}$ so that there is a $q \in \Lambda_+$ so that $f^n(q)$ lies in the region s_n for all $n \geq 0$.

(c) Prove that the region D contains a fixed point.

(d) Describe the local stable and unstable manifolds of the point found in part (c). (Do not attempt to describe the global stable and unstable manifolds.)

7. (The Solenoid) Let $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ be the circle with coordinates measured modulo 2π. Let $B = \{(x, y) : x^2 + y^2 \leq 1\}$ be the unit disk in the plane, and define $D = S^1 \times B$. Consider the map $F : D \to D$ defined by

$$F(\theta, p) = (2\theta, \frac{1}{10}p + \frac{1}{2}e^{i\theta}),$$

where $e^{i\theta} = (\cos\theta, \sin\theta)$. The map F is injective and $F(D) \subset \mathit{int}\ D$. The *solenoid* is

$$\Lambda = \bigcap_{j=0}^{\infty} F^j(D).$$

Let Σ be the inverse limit space for the doubling map $g(\theta) = 2\theta$. That is,

$$\Sigma = \{\theta = (\theta_0\theta_1\theta_2\dots) \ : \ \theta_j \in S^1 \text{ and } g(\theta_{j+1}) = \theta_j \text{ for all integers } j \geq 0\}.$$

(a) Define three maps satisfying the following statements:

(1) The *shift map* homeomorphism, $\sigma : \Sigma \to \Sigma$.

(2) The inverse of the shift map, $\sigma^{-1} : \Sigma \to \Sigma$.

(3) A homeomorphism $S : \Lambda \to \Sigma$ so that $S \circ F = \sigma \circ S$.

(You just need to define the maps, you do not need to prove the statements are satisfied.)

(b) Prove that the unstable set, $W^u(\mathbf{0})$, for the map σ of the point $\mathbf{0} = (000\dots)$ is given by

$$\left\{ \left([x], [\tfrac{x}{2}], [\tfrac{x}{4}], [\tfrac{x}{8}], \dots\right) \in \Sigma \ : \ x \in \mathbb{R} \right\}.$$

Here, we use $[x]$ to denote the equivalence class in $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ of the real number x.

8. (Toral Automorphisms) Let T denote the standard torus, $T = \mathbb{R}^2/\mathbb{Z}^2$. Recall that if $x, y \in \mathbb{R}$, we use $[x, y]$ to represent the corresponding point in the torus, which is an equivalence class,

$$[x, y] = \{(x + m, y + n) \ : \ m, n \in \mathbb{Z}\}.$$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2×2 matrix with integer entries and determinant ± 1. Suppose also that the matrix is hyperbolic. This guarantees that there are two real eigenvalues λ_s and λ_u with $0 < |\lambda_s| < 1$ and $|\lambda_u| > 1$. Then multiplication my A induces a hyperbolic automorphism of the torus $L : T \to T$.

(a) Prove that every point with rational coordinates is periodic.

(b) Let $p \in T$. Describe the stable and unstable sets of p.

(c) Is it true that there are a dense set of points which are homoclinic to $[0, 0]$? Explain why or why not.

(d) Sketch a proof that $L : T \to T$ is topologically transitive. (Just give the main ideas in such a proof.)