Math A4500: Midterm 2 Study Guide

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Disclaimer. This test is just a recommendation of things to study. You may be asked about things that do not appear here. I recommend ensuring that you can do all the problems in the sections covered in the book.

Definitions. You will be asked to define several terms on the test. The following is a list of terms which might appear.

Cantor set, totally disconnected, perfect, sequence space (or shift space), shift map, itinerary (or symbolic code), topological conjugacy, topologically transitive, sensitive dependence on initial conditions, chaotic, topologically semi-conjugate, C^r-distance, C^r-structurally stable

Theorems. Theorems and results from the book you should be able to prove.

- Propositions 4.4 and 4.6: Local behaviors near a hyperbolic fixed point.
- Propositions 5.1-5.3: Results about the Logistic map when $1 < \mu < 3$.
- Proposition 5.6: If $\mu > 2 + \sqrt{5}$ than Λ is a Cantor set.
- Proposition 6.6: Dynamical facts about the shift map on Σ_2.
- Theorem 9.8: The local behavior of a map near a hyperbolic fixed point is locally conjugate to its derivative.

Techniques. You should be able to use the various techniques we have developed for understanding particular dynamical systems acting on subsets of the real line. For instance, you should be able to:

- Prove that a subset of the real line is a Cantor set. This set could be constructed or dynamically determined.
- Prove dynamical results about a Shift Space. See Proposition 6.6.
- Prove that the dynamics of f on a set Λ is topologically conjugate to a shift map on a shift space.
- Prove that a system is chaotic.
- Prove two systems are topologically conjugate using fundamental domains. This was done in class, but also see example 9.4.
- Use topological conjugacy to prove dynamical statements about a map. We discussed in class how f and g are topologically conjugate then they have similar dynamical properties. See Theorem 7.5, for instance.
- Prove a system is C^r-structurally stable, or not C^r structurally stable for a fixed r and a fixed map f.

- Sharkovsky's theorem: You should be able to use the techniques from §1.10 to find periodic orbits for a continuous map $f : \mathbb{R} \to \mathbb{R}$.

Types of problems. There will be between four and six multi-part problems on the midterm. You'll have the full class period (100 minutes) to complete the midterm. Problems of the following forms are likely to appear:

- *Homework*: Problems similar to assigned homework problems.
- *Recall something, then prove something*: Problems that as you to recall a definition or a major result, and then use it in a proof.
- *Math comprehension:* The problem states a new definition, which may not have seen before, and asks you to use the definition in basic proofs.
- *Prove a result:* Problems which ask you to prove a result which is proved in the book. See the section titled "Theorems" above.

Practice problems. These are some problems I gave in a midterm last time I taught this course.

1. **Motivation:** Recall that we showed that when $\mu > 2 + \sqrt{5}$ that the logistic map $F_\mu(x) = \mu x(1-x)$ has an invariant Cantor set. The set was defined by

$$\Lambda = \{x \in [0,1] \ : \ F_\mu^n(x) \in [0,1] \text{ for all integers } n \geq 0\}.$$

One fact we needed to prove was that Λ is closed. This problem asks you to generalize this proof.

Prove the following statement:
Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, and let $X \subset \mathbb{R}$ be an arbitrary closed set. Prove that

$$\Lambda = \{x \in X \ : \ f^n(x) \in X \text{ for all integers } n \geq 0\}$$

is a closed set.

Solution: Here are two reasonable solutions:

Solution 1: Let (x_k) be a sequence of points in Λ which converge to some point $y \in \mathbb{R}$. We will show that $y \in \Lambda$. To do this we need to prove that for any integer n, $f^n(y) \in X$. Fix an integer n. Then because f^n is continuous,

$$f^n(y) = \lim_{k \to \infty} f^n(x_k).$$

Since each $x_k \in \Lambda$, we know that each $f^n(x_k) \in X$. Then since x_k is closed, $f^n(y) \in X$.

Solution 2: Observe that $X^c = \mathbb{R} \setminus X$ is open. Then, the following statements are equivalent:

- $f^n(x) \notin X$.
- $f^n(x) \in X^c$.
- $x \in f^{-n}(X^c)$. (Recall: $f^{-n}(X^c) = \{x \in \mathbb{R} \; : \; f^n(x) \in X^c\}$ is the inverse image of X^c under f^n.)

The set $\Lambda^c = \mathbb{R} \smallsetminus \Lambda$ is the collection of points for which there is an n so that $f^n(x) \notin X$. Thus,

$$\Lambda^c = \bigcup_{n \geq 0} f^{-n}(X^c).$$

Since the inverse image of an open set under a continuous map is open, $f^{-n}(X^c)$ is open. Then Λ^c is open, since it is a union of open sets. Since its compliment is open, Λ is closed.

2. (a) (10 points) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a homeomorphism so that $f(x) > x$ for all $x \in \mathbb{R}$. Prove that $\lim_{n\to\infty} f^n(x) = +\infty$ for all $x \in \mathbb{R}$.

Solution: Let $x \in \mathbb{R}$ be arbitrary. Let $x_n = f^n(x)$ for $n \geq 0$. This defines a sequence. Since $x_{n+1} = f(x_n)$ for all n, we know by assumption that $x_{n+1} > x_n$. Thus, the sequence (x_n) is monotone increasing. Then it converges to a value in $\mathbb{R} \cup \{+\infty\}$. We want to show it converges to $+\infty$. So, suppose to the contrary that

$$\lim_{n\to\infty} x_n = L \in \mathbb{R}.$$

Then by continuity of f, we know that

$$f(L) = f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = L.$$

So, L is a fixed point of f. But this contradicts the statement that $f(x) > x$ for all x.

(b) (6 points) Complete the following definition:
Let $f : A \to A$ and $g : B \to B$ be two maps with $A, B \subset \mathbb{R}$. We say f and g are *topologically conjugate* if ...

Solution: There is a homeomorphism $\phi : A \to B$ so that $\phi \circ f(a) = g \circ \phi(a)$ for all $a \in A$.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be a homeomorphism so that $f(x) > x$ for all $x \in \mathbb{R}$ be as in the prior part, and let $g : \mathbb{R} \to \mathbb{R}$ be $g(x) = x + 1$. Prove that f and g are topologically conjugate.
(*Hint:* Use part (a) and the similar fact that $\lim_{n\to\infty} f^{-n}(x) = -\infty$ for all $x \in \mathbb{R}$. This second fact can be assumed without proof.)

Solution: We will construct a homeomorphism $\phi : \mathbb{R} to \mathbb{R}$ so that

$$\phi \circ f(x) = g \circ \phi(x) \quad \text{for all } x \in \mathbb{R}.$$

Let $x_0 \in \mathbb{R}$ be arbitrary. Define $x_n = f^n(x)$ for all $n \in \mathbb{Z}$. Then we have $x_{n+1} > x_n$ for all $n \in \mathbb{Z}$. Also from part (a) and the hint, we have

$$\lim_{n\to\infty} x_{-n} = -\infty \quad \text{and} \quad \lim_{n\to\infty} x_n = \infty.$$

For each n define the half open interval $I_n = [x_n, x_{n+1})$. This is a disjoint set of intervals that cover all of $\mathbb{R}$. Also $f^k(I_n) = I_{n+k}$ for all $n, k \in \mathbb{Z}$.
Now let $\phi_0 : I_0 \to [0, 1)$ be an arbitrary homeomorphism. We extend it to be a topological conjugacy by defining

$$\phi(y) = n + \phi_0 \circ f^{-n}(y) \quad \text{if } y \in I_n.$$

Remark: I would have been happy if you stopped at the is point.
This is a well defined function since the intervals I_n are pairwise disjoint and cover $\mathbb{R}$. Observe that ϕ is continuous at y if y lies in the interior of an interval I_n, since both ϕ_0 and f^{-n} are continuous. If y is not in the interior of an I_n, then $y = x_n$ for some n. We will now check continuity at x_n. Note that $\phi(x_n) = n$. When we approach x_n from the right, we eventually land in I_n. So,

$$\lim_{t\to x_n^+} \phi(t) = \lim_{t\to x_n^+} n + \phi_0 \circ f^{-n}(t) = n + \phi_0 \circ f^{-n}(x_n) = n + \phi_0(x_0) = n.$$

When t approaches x_n from the left, we eventually have $t \in I_{n-1}$, and $f^{-n+1}(t)$ approaches x_1 from the left. Since $\phi_0 : [x_0, x_1) \to [0, 1)$ is a homeomorphism, we have:

$$\lim_{t\to x_n^-} \phi(t) = \lim_{t\to x_n^-} n - 1 + \phi_0 \circ f^{-n+1}(t) = \lim_{s\to x_1^-} n - 1 + \phi_0(s) = n - 1 + 1 = n.$$

This proves the limit from the left and right coincide with the value $\phi(x_n) = n$, so ϕ is continuous at x_n.
To see ϕ is invertible, not that the inverse is

$$y \mapsto f^{\lfloor y \rfloor} \circ \phi_0^{-1}(y - \lfloor y \rfloor),$$

where $\lfloor y \rfloor$ denotes the greatest integer less than or equal to y. A similar argument can be used to show that this inverse map is continuous. (Actually a continuous bijection from $\mathbb{R}$ to $\mathbb{R}$ always has a continuous inverse.)
To see that ϕ is a topological conjugacy it must be checked that $\phi \circ f = g \circ \phi$. Pick a $y \in \mathbb{R}$. Then $y \in I_n$ for some $n \in \mathbb{Z}$, so by definition of ϕ and g,

$$g \circ \phi(y) = g\Big(n + \phi_0 \circ f^{-n}(y)\Big) = n + 1 + \phi_0 \circ f^{-n}(y).$$

Since $y \in I_n$, we have $f(y) \in I_{n+1}$ and by definition of ϕ,

$$\phi\big(f(y)\big) = n + 1 + \phi_0 \circ f^{-n-1}\big(f(y)\big) = n + 1 + \phi_0 \circ f^{-n}(y).$$

These quantities are equal, so we have proved the desired equation.

Recall the following definitions and use them in the following proofs.

- The C^r-*distance* between two C^r functions $f, g : \mathbb{R} \to \mathbb{R}$ is

$$d_r(f, g) = \sup_{x\in\mathbb{R}} \max_{k=0,\dots,r} |f^{(k)}(x) - g^{(k)}(x)|.$$

- A C^r function $g : \mathbb{R} \to \mathbb{R}$ is C^r-*structurally stable* if there is an $\epsilon > 0$ so that f and g are topologically conjugate whenever f is a C^r function satisfying $d_r(f, g) < \epsilon$.

(d) Prove that $g(x) = x + 1$ is C^1-structurally stable. (*Hint:* You can use part (c), even if you were not successful in writing a proof for that part.)

Solution: Suppose that $d_1(f, g) < \frac{1}{2}$. Then for all x, $|f'(x) - g'(x)| < \frac{1}{2}$ and $\frac{1}{2} < f'(x) < \frac{3}{2}$. Thus, f is an orientation preserving homeomorphism. Also, $|f(x) - g(x)| < \frac{1}{2}$. So,

$$x + \frac{1}{2} < f(x) < x + \frac{3}{2}.$$

In particular $f(x) > x$ for all x. Then it follows from part (c) that f and g are topologically conjugate.

(e) Prove that $g(x) = x + 1$ is not C^0-structurally stable.

Solution: Suppose to the contrary that g is C^0 structurally stable. Then there is an $\epsilon > 0$ so that $|f(x) - g(x)| < \epsilon$ for all $x \in \mathbb{R}$ implies that f and g are topologically conjugate. We will construct a counterexample. Let

$$f(x) = x + 1 - \frac{\epsilon}{2} \sin(\frac{x\pi}{\epsilon}).$$

Then $d_0(f, g) = \frac{\epsilon}{2}$. But,

$$f(0) = 1 \quad \text{and} \quad f(\frac{\epsilon}{2}) = \frac{\epsilon}{2} + 1 - \frac{\epsilon}{2} \sin(\frac{\pi}{2}) = 1.$$

This reveals a contradiction. There are at least two solutions to $f(0) = 1$, but there is only one solution to $g(x) = b$ for every b. So f and g can not be topologically conjugate.

3. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and has the following orbit of period four:

$$f(0) = 2, \quad f^2(0) = 3, \quad f^3(0) = 1, \quad f^4(0) = 0.$$

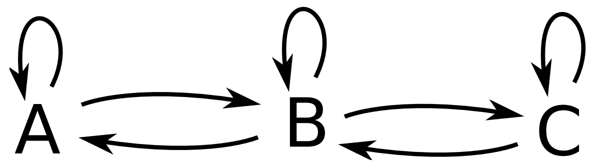
Define the the intervals $A = [0, 1]$, $B = [1, 2]$ and $C = [2, 3]$.

(a) Prove that f has a point of least period 13 which stays within $A \cup B \cup C$. Find an explicit sequence of intervals visited by such an orbit.

Solution: Observe the following covering relations:

- $[0, 2] \subset f(A)$, so $f(A)$ covers A and B.
- $[0, 3] \subset f(B)$, so $f(B)$ covers A, B and C.
- $[1, 3] \subset f(C)$, so $f(C)$ covers B and C.

These covering relations can be summarized by the graph below:



Here an arrow goes from an interval I to J if $f(I)$ covers J (i.e. $J \subset f(I)$). Consider the loop of length 13 in the graph:

$$C \to B \to A \to \ldots \to A \to B \to C.$$

Then the lemmas used in the proofs of Sarkovskii's Theorem imply that there is a point $x \in C = [2, 3]$ so that:

- $f(x) \in B$.
- $f^k(x) \in A$ for $k = 2, \ldots, 11$.
- $f^{12}(x) = B$.
- $f^{13}(x) = x \in C$.

This point x has period thirteen. We must argue that this is the least period. Since x is prime, if it is not least period thirteen, then it is a fixed point. But x can not be fixed since $x \in C$ and $f^2(x) \in A$ and $A \cap C = \emptyset$.

(b) Use Sharkovsky's theorem to prove that f has points of all possible periods. (*Remarks:* It is easier to use Sharkovsky's theorem then to prove this fact with the methods used to prove Sharkovsky's theorem, however the application of Sharkovsky's theorem is indirect. You can use the lemmas from the proof of Sharkovsky's theorem without proving the lemmas.)

Solution: Consider the path of length three:

$$A \to B \to B \to A.$$

Lemmas used to prove Sarkovskii's Theorem imply that there is a point $x \in A$ so that $f(x) \in B$, $f^2(x) \in B$, and $f^3(x) = x \in A$. We claim that such a point has least period three. Since three is prime, if this is false then x would have to be fixed by f. Suppose $f(x) = x$. Then since $x \in A$ and $f(x) \in B$, we must have $x \in A \cap B = \{1\}$. So $x = 1$. But, $x = 1$ has period four, which is a contradiction. Thus, x has has least period three.

Then by Sarkovskii's Theorem, x has points of all possible periods.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable function with finitely many critical points. Prove that $f^m(x)$ also has only finitely many critical points for every $m \geq 1$.

Solution: Suppose that f has k critical points. Observe that each point in $\mathbb{R}$ has at most $k+1$ preimages under f. This, is because between every two preimages of a point there must be a critical point (by Rolle's Theorem).

We prove that $f^m(x)$ has only finitely many critical points by induction in $m \geq 1$. The case of $m = 1$ is true by assumption: f has only k critical points. Now suppose that $f^m(x)$ has K critical points. We will prove that $f^{m+1}(x)$ has only finitely many critical points. By the chain rule

$$(f^{m+1})'(x) = (f^m)'\big(f(x)\big)f'(x).$$

So, x is a critical point for f^{m+1} if and only if $f(x)$ is a critical point for f^m or x is a critical point for f. Since f is at most $k+1$-to-one, there are at most $(k+1)K$ points so that $f(x)$ is a critical point for f^m. Also, there are k critical points for f. So, there are at most $(k+1)K + k$ critical points for f^{m+1}. This is a finite number as desired.

5. (The Logisitic Map) Let $F(x) = \mu x(1-x)$ with $\mu > 2 + \sqrt{5}$. Let

$$\Lambda = \{x \ : \ F_\mu^n(x) \in [0,1] \text{ for all } n \geq 0\}.$$

Let $p_0 \in (0, \frac{1}{2})$ and $p_1 \in (\frac{1}{2}, 1)$ be the points so that $f(p_0) = f(p_1) = 1$. Define $I_0 = [0, p_0]$ and $I_1 = [p_1, 1]$. Then, $\Lambda \subset I_0 \cup I_1$, and $|F'(x)| > k$ for some $k > 1$ and all $x \in I_0 \cup I_1$.

(a) (5 points) Define *sensitive dependence on initial conditions.*

(b) (5 points) Describe how to code the orbit of a point in Λ using a one-sided shift space. That is, describe the map $s : \Lambda \to \Sigma_2$.

(c) (5 points) Prove that the coding map s is injective.

(d) (5 points) Assuming that the coding map s is a topological conjugacy between $F|_\Lambda : \Lambda \to \Lambda$ and $\sigma : \Sigma_2 \to \Sigma_2$, prove that F has a dense set of periodic points. (You do not need to prove that σ has a dense set of periodic points.)

6. (Shift spaces) Let $\mathcal{A} = \{0, 1\}$. Define the one-sided shift space

$$\Sigma = \{(s_0 s_1 s_2 \ldots) \ : \ s_j \in \mathcal{A} \text{ for all integers } j \geq 0\}.$$

The shift space becomes a metric space when equipped with either of the following distance functions:

$$d(\mathbf{s},\mathbf{t}) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i} \quad \text{or} \quad d'(\mathbf{s},\mathbf{t}) = \begin{cases} 2^{-n} & \text{if } \mathbf{s} \neq \mathbf{t} \text{ and } n = \min\{i : s_i \neq t_i\} \\ 0 & \text{iff } \mathbf{s} = \mathbf{t}. \end{cases}$$

Both d and d' determine the same topology. The shift map $\sigma : \Sigma \to \Sigma$ is defined by

$$\sigma(s_0 s_1 s_2 \ldots) = (s_1 s_2 s_3 \ldots).$$

Use the definitions above and either of the metrics to answer the following questions.

(a) (5 points) Is the map σ a homeomorphism? Why or why not?

(b) (5 points) Prove that σ-periodic points are dense in Σ.

(c) (5 points) Define *topologically transitive*.

(d) (5 points) Prove that the shift map is topologically transitive on Σ.