Math A4500: Midterm 1 Study Guide

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Disclaimer. This test is just a recommendation of things to study. You may be asked about things that do not appear here. I recommend ensuring that you can do all the problems in the sections covered in the book.

Definitions. You will be asked to define several terms on the test. The following is a list of terms which might appear.

class C^r, one-to one, onto, homeomorphism, C^r-diffeomorphism, limit point, closed set, open set, dense, forward orbit, backward orbit, orbit, periodic point, period, prime period (or least period), eventually periodic, forward and backward asymptotic, critical point, hyperbolic periodic point, multiplier, attracting periodic point, repelling periodic point

You are expected to know the definitions given in the book. However, a better definition of dense is a subset of $U \subset S$ is *dense* in S if $S \subset \bar{U}$. Maps $f : X \to X$ and $g : Y \to Y$ are *topologically semi-conjugate* if there is a continuous map $h : X \to Y$ so that $h \circ f(x) = g \circ h(x)$ for all $x \in X$. The maps are *topologically conjugate* if h can be taken to be a homeomorphism.

Advanced Calculus. You should be able to use the state the basic theorems from calculus and be able to use them to prove dynamical results. The following are such theorems:

Intermediate value theorem, Mean value theorem, Implicit function theorem.

Theorems. Theorems and results from the book you should be able to prove.

- Proposition 2.11: If I is a closed and bounded interval, and $f : I \to I$ is continuous, then f has a fixed point.
- Proposition 2.12: If $f : I \to I$ is a continuous map of a closed and bounded interval I, and $|f'(x)| < 1$, then f has a unique fixed point and f contracts distances.
- Propositions 4.4 and 4.6: Local behaviors near a hyperbolic fixed point.

Techniques. You should be able to use the various techniques we have developed for understanding particular dynamical systems acting on subsets of the real line. For instance, you should be able to:

- Find fixed and periodic points of maps of $\mathbb{R}$ and S^1.
- Understand the long term behaviors of points (as in the term *forward* and *backward asymptotic.*
- Use topological conjugacy to prove dynamical statements about a map. We discussed in class how f and g are topologically conjugate then they have similar dynamical properties.

Types of problems. There will be between four and six multi-part problems on the midterm. You'll have the full class period (100 minutes) to complete the midterm. Problems of the following forms are likely to appear:

- *Homework*: Problems similar to assigned homework problems.
- *Recall Something, then prove something.*: Problems that as you to recall a definition or a major result, and then use it in a proof.
- *Math comprehension:* The problem states a new definition, which may not have seen before, and asks you to use the definition in basic proofs.
- *Prove a result:* Problems which ask you to prove a result which is proved in the book. See the section titled "Theorems" above.

Practice problems. These are some problems I gave in a midterm last time I taught this course.

1. Complete the following definitions:

   (a) Let $r \geq 0$ be an integer. A real valued function f defined on $X \subset \mathbb{R}$ is *of class* C^r if...

   Solution: The function f is r-times differentiable, and $f^{(r)}$ is continuous.

   (b) Let $X \subset \mathbb{R}$ and let $f : X \to X$ be a function. Also let $p \in X$ be a point of prime period n under f. Then, a point $x \in X$ is *forward asymptotic* to p if ...

   Solution: $\lim_{k\to\infty} f^{kn}(x) = p$.

2. Let $f : [a, b] \to [a, b]$ be a continuous function defined on a closed interval with $a < b$. Prove that f has a fixed point.

   Solution: If $f(a) = a$ or $f(b) = b$, we have a fixed point. So, suppose that $f(a) \neq a$ and $f(b) \neq b$. Then since $f(a)$ and $f(b)$ both lie in $[a, b]$, we know that $f(a) > a$ and $f(b) < b$. Now let $g(x) = f(x) - x$. This function is continuous because f is continuous. Also we have $g(a) > 0$ and $g(b) < 0$. So, by the Intermediate Value Theorem, there is a point $c \in (a, b)$ so that $g(c) = 0$. For this c, we have $f(c) = c$.

3. (a) Complete the following definition:
   Let $p \in \mathbb{R}$ be a point of prime period n under a real-valued function f. The point p is *hyperbolic* if ...

Solution: $|(f^n)'(p)| \neq 1$.

(b) Let $f : [a, b] \to [a, b]$ be a C^1 function, and suppose that p is a hyperbolic fixed point with $|f'(p)| < 1$. Prove that there is an open interval U about p such that if $x \in U$, then

$$\lim_{n\to\infty} f^n(x) = p.$$

Solution: Suppose that $f(p) = p$ and $|f'(p)| < 1$. Choose a real number λ satisfying $|f'(p)| < \lambda < 1$. Since f is C^1, we know that f' is continuous. Let $\epsilon = \lambda - |f'(p)| > 0$. By continuity of f', there is a δ so that

$$|x - p| < \delta \quad \text{implies} \quad |f'(p) - f'(x)| < \epsilon. \tag{1}$$

Let $U = (p - \delta, p + \delta)$. Note that the conclusion above can be written as

$$-\lambda \leq f'(p) - \epsilon < f'(x) < f'(p) + \epsilon \leq \lambda.$$

So we see that

$$x \in U \quad \text{implies} \quad |f'(x)| < \lambda. \tag{2}$$

Let $U = (p - \delta, p + \delta)$. We claim that for any $x \in U$, we have

$$|f(x) - p| < \lambda|x - p|. \tag{3}$$

Choose any $x \in U$. Then by the Mean Value Theorem, there is a point c between p and x so that

$$f'(c) = \frac{f(x) - f(p)}{x - p}.$$

Since $p \in U$ and $x \in U$, we know that $c \in U$. So,

$$\frac{|f(x) - f(p)|}{|x - p|} = |f'(c)| < \lambda.$$

This is equivalent to equation 3.

Now choose any $x \in U$. Let $n \geq 0$ be an integer. Consider the following statements:

(1_n). $f^n(x) \in U$.

(2_n). $|f^{n+1}(x) - p| < \lambda|f^n(x) - p|$.

Since $x \in U$, statement (1_0) holds. Equation 3 gives us that statement (1_n) implies statement (2_n). If (1_n) and (2_n) hold, then $f^n(x)$ is within δ of p and

$f^{n+1}(x)$ is even closer, so statement (1_{n+1}) holds. So, the statements hold for all n by induction. In particular, we have

$$|f^n(x) - p| < \lambda^n |x - p| \to 0 \quad \text{as } n \to \infty.$$

So, $f^n(x)$ tends to p as desired.

Remark: The inductive argument here is more detailed than what I expected from you.

4. (a) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a homeomorphism so that $f(x) > x$ for all $x \in \mathbb{R}$. Prove that $\lim_{n\to\infty} f^n(x) = +\infty$ for all $x \in \mathbb{R}$.

Solution: Let $x \in \mathbb{R}$ be arbitrary. Let $x_n = f^n(x)$ for $n \geq 0$. This defines a sequence. Since $x_{n+1} = f(x_n)$ for all n, we know by assumption that $x_{n+1} > x_n$. Thus, the sequence (x_n) is monotone increasing. Then it converges to a value in $\mathbb{R} \cup \{+\infty\}$. We want to show it converges to $+\infty$. So, suppose to the contrary that

$$\lim_{n\to\infty} x_n = L \in \mathbb{R}.$$

Then by continuity of f, we know that

$$f(L) = f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = L.$$

So, L is a fixed point of f. But this contradicts the statement that $f(x) > x$ for all x.

(b) Complete the following definition:
Let $f : A \to A$ and $g : B \to B$ be two maps with $A, B \subset \mathbb{R}$. We say f and g are *topologically conjugate* if ...

Solution: There is a homeomorphism $\phi : A \to B$ so that $\phi \circ f(a) = g \circ \phi(a)$ for all $a \in A$.

(c) (10 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a homeomorphism so that $f(x) > x$ for all $x \in \mathbb{R}$ be as in the prior part, and let $g : \mathbb{R} \to \mathbb{R}$ be $g(x) = x + 1$. Prove that f and g are topologically conjugate.
(*Hint:* Use part (a) and the similar fact that $\lim_{n\to\infty} f^{-n}(x) = -\infty$ for all $x \in \mathbb{R}$. This second fact can be assumed without proof.)

Solution: We will construct a homeomorphism $\phi : \mathbb{R} \to \mathbb{R}$ so that

$$\phi \circ f(x) = g \circ \phi(x) \quad \text{for all } x \in \mathbb{R}.$$

Let $x_0 \in \mathbb{R}$ be arbitrary. Define $x_n = f^n(x)$ for all $n \in \mathbb{Z}$. Then we have

$x_{n+1} > x_n$ for all $n \in \mathbb{Z}$. Also from part (a) and the hint, we have

$$\lim_{n\to\infty} x_{-n} = -\infty \quad \text{and} \quad \lim_{n\to\infty} x_n = \infty.$$

For each n define the half open interval $I_n = [x_n, x_{n+1})$. This is a disjoint set of intervals that cover all of $\mathbb{R}$. Also $f^k(I_n) = I_{n+k}$ for all $n, k \in \mathbb{Z}$.

Now let $\phi_0 : I_0 \to [0, 1)$ be an arbitrary homeomorphism. We extend it to be a topological conjugacy by defining

$$\phi(y) = n + \phi_0 \circ f^{-n}(y) \quad \text{if } y \in I_n.$$

Remark: I would have been happy if you stopped at the is point.

This is a well defined function since the intervals I_n are pairwise disjoint and cover $\mathbb{R}$. Observe that ϕ is continuous at y if y lies in the interior of an interval I_n, since both ϕ_0 and f^{-n} are continuous. If y is not in the interior of an I_n, then $y = x_n$ for some n. We will now check continuity at x_n. Note that $\phi(x_n) = n$. When we approach x_n from the right, we eventually land in I_n. So,

$$\lim_{t\to x_n^+} \phi(t) = \lim_{t\to x_n^+} n + \phi_0 \circ f^{-n}(t) = n + \phi_0 \circ f^{-n}(x_n) = n + \phi_0(x_0) = n.$$

When t approaches x_n from the left, we eventually have $t \in I_{n-1}$, and $f^{-n+1}(t)$ approaches x_1 from the left. Since $\phi_0 : [x_0, x_1) \to [0, 1)$ is a homeomorphism, we have:

$$\lim_{t\to x_n^-} \phi(t) = \lim_{t\to x_n^-} n - 1 + \phi_0 \circ f^{-n+1}(t) = \lim_{s\to x_1^-} n - 1 + \phi_0(s) = n - 1 + 1 = n.$$

This proves the limit from the left and right coincide with the value $\phi(x_n) = n$, so ϕ is continuous at x_n.

To see ϕ is invertible, not that the inverse is

$$y \mapsto f^{\lfloor y \rfloor} \circ \phi_0^{-1}(y - \lfloor y \rfloor),$$

where $\lfloor y \rfloor$ denotes the greatest integer less than or equal to y. A similar argument can be used to show that this inverse map is continuous. (Actually a continuous bijection from $\mathbb{R}$ to $\mathbb{R}$ always has a continuous inverse.)

To see that ϕ is a topological conjugacy it must be checked that $\phi \circ f = g \circ \phi$. Pick a $y \in \mathbb{R}$. Then $y \in I_n$ for some $n \in \mathbb{Z}$, so by definition of ϕ and g,

$$g \circ \phi(y) = g\Big(n + \phi_0 \circ f^{-n}(y)\Big) = n + 1 + \phi_0 \circ f^{-n}(y).$$

Since $y \in I_n$, we have $f(y) \in I_{n+1}$ and by definition of ϕ,

$$\phi\big(f(y)\big) = n + 1 + \phi_0 \circ f^{-n-1}\big(f(y)\big) = n + 1 + \phi_0 \circ f^{-n}(y).$$

These quantities are equal, so we have proved the desired equation.