Math A4500: Dynamical Systems: First Midterm Solutions

Tuesday, Feb 28th, 2017

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1. Consider the Logistic map $F(x) = 4x(1-x)$.

   (a) (8 points) Find all fixed points of F.

   Solution: Fixed points are solutions to $F(x) = x$. By simplifying we see

   $$F(x) = 4x(1-x) = 4x - 4x^2.$$

   So the fixed points are the roots of the polynomial $3x - 4x^2$. We can factor this as $x(3-4x)$ so the roots are $x = 0$ and $x = \frac{3}{4}$. These are also the fixed points.

   (b) (10 points) Find the periodic points of prime period 2 under F. (*Hint:* This is a bit painful but the first part can help a bit...)

   Solution: Period two points are solutions to $F^2(x) = x$. We have

   $$\begin{aligned} F^2(x) &= F(4x-4x^2) = 4(4x-4x^2) - 4(4x-4x^2)^2 \\ &= 16x - 16x^2 - 64x^2 + 128x^3 - 64x^4 \\ &= 16x - 80x^2 + 128x^3 - 64x^4. \end{aligned}$$

   This means that the period two points are roots of

   $$F^2(x) - x = 15x - 80x^2 + 128x^3 - 64x^4.$$

   We can clearly factor out x, and this makes sense because fixed points are also period 2 points (just not of prime period 2). This means that $x = \frac{3}{4}$ should also be a root and that $4x - 3$ should also be a factor of this polynomial. We simplify:

   $$F^2(x) - x = x(15 - 80x + 128x^2 - 64x^3) = x(3-4x)(5 - 20x + 16x^2).$$

   Then we can find the remaining roots using the quadratic formula:

   $$x_\pm = \frac{20 \pm \sqrt{80}}{32} = \frac{5 \pm \sqrt{5}}{8}.$$

   Since these points are not fixed, they must be points of prime period 2. This also means that these two points form a periodic orbit of F.

(c) (6 points) Decide if the period 2 orbit found in the previous part is attracting, repelling or neutral.

> **Solution:** We must compute the multiplier of the orbit which from the chain rule is
> $$(F^2)'(x_\pm) = F'(x_+)F'(x_-).$$
> Observe that $F'(x) = 4 - 8x$. So we compute
> $$F'(x_\pm) = 4 - (5 \pm \sqrt{5}) = -1 \mp \sqrt{5}.$$
> Then
> $$(F^2)'(x_\pm) = F'(x_+)F'(x_-) = (-1 - \sqrt{5})(-1 + \sqrt{5}) = 1 - 5 = -4.$$
> Since $|(F^2)'(x_\pm)| > 1$, this periodic orbit is repelling.

2. (a) (8 points) Complete the following definition:
Let $f : \mathbb{R} \to \mathbb{R}$ and p be a periodic point of period n under f. A point $x \in \mathbb{R}$ is *forward asymptotic* to p if ...

> **Solution:**
> $$\lim_{k \to +\infty} f^{nk}(x) = p.$$

(b) (12 points) Let $I = [a, b] \subset \mathbb{R}$ be an interval with $a < b$. Suppose that $f : I \to I$ is continuous and satisfies:

(1) $f(a) = a$.

(2) $f(b) = b$.

(3) $f(x) < x$ for all x satisfying $a < x < b$.

Prove that any x satisfying $a < x < b$ is forward asymptotic to a.

> **Solution:** Suppose $a < x < b$. Consider the orbit $f^n(x)$ as a sequence. By (3), this sequence is strictly decreasing. Also the sequence is bounded from below by a. Therefore there is a limit $y = \lim_{n\to\infty} f^n(x)$. Since f is continuous we know that
> $$f(y) = \lim_{n \to \infty} f \circ f^n(y).$$
> But the above is the same sequence (with the first term removed). Therefore we have $f(y) = y$. But in light of (3) we must have $y = a$, so $\lim f^n(x) = a$ as desired.

3. (a) (8 points) Complete the following definition:

If $f : \mathbb{R} \to \mathbb{R}$ is differentiable and x is a point of prime (or least) period p under f, then the *multiplier* of x is ...

Solution: $(f^p)'(x)$.

(b) (16 points) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be continuous. We say h is a *topological semiconjugacy* from f to g if $h : \mathbb{R} \to \mathbb{R}$ is a homeomorphism and satisfies $h \circ f = g \circ h$ on $\mathbb{R}$. Prove that if h is a topological semiconjugacy from f to g and $x \in \mathbb{R}$ is a periodic point of f of period p, then $y = h(x)$ is a periodic point of g of period p.

Solution: Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be of class C^1. Suppose h is a C^1 conjugacy from f to g and that $x \in \mathbb{R}$ is a periodic point of f of period p. Let $y = h(x)$.
We claim that y is periodic under g with period p. To see this observe that

$$g^p(y) = g^p \circ h(x) = h \circ f^p(x) = h(x) = y.$$

The step $g^p \circ h(x) = h \circ f^p(x)$ can be seen by induction. It is true that $g^k \circ h = h \circ f^k$ for all k. This is the definition of semi-conjugacy for $k = 1$ (our base case). Then assuming $g^k \circ h = h \circ f^k$ we have

$$g^{k+1} \circ h = g \circ g^k \circ h = g \circ h \circ f^k = h \circ f \circ f^k = h \circ f^{k+1},$$

which is the inductive step.

(c) Continuing (b) prove that if f, g and h are differentiable and $h'(x) \neq 0$ for all x, then $(f^p)'(x) = (g^p)'(y)$.

Solution: This is a consequence of the formula that

$$g^p \circ h(x) = h \circ f^p(x).$$

By applying the chain rule, we have

$$\left[(g^p)'\big(h(x)\big)\right] h'(x) = h'\big(f^p(x)\big) \left[(f^p)'(x)\right].$$

Since x is period p, we know $f^p(x) = x$. The right side simplifies as $h'(x) \left[(f^p)'(x)\right]$. We can divide through by $h'(x)$ since this quantity is non-zero yielding

$$(g^p)'\big(h(x)\big) = (f^p)'(x).$$

4. (a) (10 points) State the Mean Value Theorem.

Solution: Suppose that f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there is a point $c \in (a, b)$ so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

(b) (12 points) Let $f : \mathbb{R} \to \mathbb{R}$ be of class C^1 and suppose that $|f'(x)| < 1$ for all $x \in \mathbb{R}$. Prove that f can have no more than one fixed point.

Solution: Suppose to the contrary that f has distinct fixed points p and q. Then $f(p) = p$ and $f(q) = q$. We can assume without loss of generality that $p < q$. By the Mean Value Theorem there is a point $c \in (p, q)$ so that

$$f'(c) = \frac{f(q) - f(p)}{q - p} = \frac{q - p}{q - p} = 1.$$

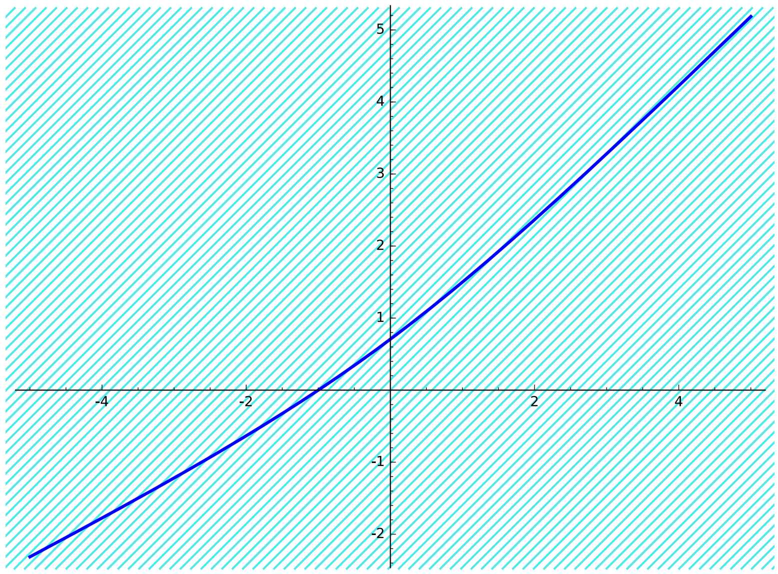
But this is a contradiction since $|f'(x)| < 1$ for all $x \in \mathbb{R}$.

(c) (10 points) Give an example of a C^1 diffeomorphism $f : \mathbb{R} \to \mathbb{R}$ so that

$$0 < f'(x) < 1 \quad \text{for all } x \in \mathbb{R}$$

but f has no fixed points. Prove your answer is correct.

Solution: **Remarks:** The graph of f must avoid the diagonal line $y = x$. The fact that $f'(0) > 0$ means that f should be increasing. Graphically, the statement that $f'(x) < 1$ means that we are never tangent to a line of slope 1. Instead the graph moves rightward over the slope 1 lines as x increases. The graph for Solution 1 is shown with the diagonal below:



Solution 1: A simple example is given by a hyperbola with asymptotes $y = x$ and another line with slope between zero and one. We'll use $2y = x$. Let $F(x, y) = (y - x)(2y - x)$. We define our function $y = f(x)$ implicitly by the rules that

$$(y - x)(2y - x) = 1, \quad y > x \quad \text{and} \quad 2y > x.$$

(This picks out the top piece of a hyperbola. You can get an explicit solution using the quadratic formula.) To see that this determines a function observe that $F(x, \max\{x, \frac{x}{2}\}) = 0$. Both the functions $y \mapsto y - x$ and $y \mapsto 2y - x$ are monotone increasing in y (with x fixed) and tend toward $+\infty$ as $y \to +\infty$. By the intermediate value theorem there must be a $y > \max\{x, \frac{x}{2}\}$ so that $F(x, y) = 1$. This proves that f is well defined as a function. Also since $y > x$ on all of the graph we know that f has no fixed points.

It remains to check that $0 < f'(x) < 1$ at all $x \in \mathbb{R}$. Observe using the product rule that $\nabla F(x, y) = \big(-(2y - x) - (y - x), (2y - x) + 2(y - x)\big)$. By the implicit function theorem, if $y = f(x)$ we have

$$f'(x) = \frac{-F_x(x, y)}{F_y(x, y)} = \frac{(2y - x) + (y - x)}{(2y - x) + 2(y - x)}.$$

All the terms in the parenthesis are positive as noted above so $f'(x) > 0$. Also setting $t = (2y - x) + (y - x) > 0$ we see

$$f'(x) = \frac{t}{t + (y - x)} < 1 \quad \text{since } y - x > 0.$$

(Actually you can show using a similar method that $f'(x) > \frac{1}{2}$ for all x.)