

# CANTOR SETS IN THE LOGISTIC FAMILY

W. PATRICK HOOPER

We consider the logistic map  $F_\mu(x) = \mu x(1 - x)$  when  $\mu > 4$ . Define

$$\Lambda = \{x \in [0, 1] : F_\mu^n(x) \in [0, 1] \text{ for all } n \geq 0\}.$$

We will prove the following theorem:

**Theorem 1.** *If  $\mu > 2 + \sqrt{5}$ , then  $\Lambda$  is a Cantor set.*

The goal of this note is to give a more explicit proof of this result than what is offered in Devaney's book. The proof that  $\Lambda$  is perfect is different than what we did in class, and basically just is a more detailed account of Devaney's proof.

We will now give an equivalent description of  $\Lambda$ . Define the set

$$A_0 = \{x \in [0, 1] : F_\mu(x) \notin [0, 1]\}.$$

Then  $A_0$  is an open interval:

$$A_0 = \left( \frac{1}{2} - \frac{\sqrt{\mu - 4}}{2\sqrt{\mu}}, \frac{1}{2} + \frac{\sqrt{\mu - 4}}{2\sqrt{\mu}} \right).$$

We can then inductively define  $A_n$  for all integers  $n \geq 0$  inductively according to the rule that  $A_{n+1} = F_\mu^{-1}(A_n)$ . Then we have

$$\Lambda = [0, 1] \setminus \bigcup_{n=0}^{\infty} A_n.$$

The reason for taking  $\mu > 2 + \sqrt{5}$  is given by the following, which is not hard to prove.

**Proposition 2.** *If  $\mu > 2 + \sqrt{5}$ , then  $|F'_\mu(x)| > 1$  for all  $x \in [0, 1] \setminus A_0$ .*

The relevance of the derivative is given by the following, which follows from the fundamental theorem of calculus and was proved in class.

**Proposition 3.** *If  $f$  is a real-valued function which is differentiable and monotone on an interval  $J \subset \mathbb{R}$ , then*

$$\text{length } f(J) = \int_J |f'(x)| dx.$$

These two statements are enough to prove our Theorem.

*Proof of Theorem 1.* Suppose that  $\mu > 2 + \sqrt{5}$ . We will prove that  $\Lambda$  is closed, totally disconnected, and perfect.

1. We will prove that  $\Lambda$  is closed. Observe that  $A_0$  is open. Recall that  $A_{n+1} = F_\mu^{-1}(A_n)$ . Since  $F_\mu$  is continuous,  $F_\mu^{-1}(U)$  is open for every open set  $U \subset \mathbb{R}$ . Therefore, we can see by induction that  $A_n$  is open for all  $n$ . Then  $\Lambda = [0, 1] \setminus \bigcup_{n=0}^{\infty} A_n$  must be closed.

**2.** We will prove that  $\Lambda$  is totally disconnected, that is  $\Lambda$  does not contain any (non-degenerate) intervals. By proposition 2, there is a  $\lambda > 1$  so that  $|F'_\mu(x)| \geq \lambda$  for all  $x \in [0, 1] \setminus A_0$ .

Suppose  $\Lambda$  does contain an interval  $J_0$  of positive length. Then, inductively define  $J_n$  for all  $n \geq 0$  according to the rule that  $J_{n+1} = F_\mu(J_n)$ . This makes  $J_n = F_\mu^n(J_0)$  for all  $n$ . Since  $J_0 \subset \Lambda$ , we know that  $J_n \subset \Lambda$  for all  $n$ . Then, it must be that each  $J_n$  lies in one of the two closed intervals making up  $[0, 1] \setminus A_0$ . Since  $F_\mu$  is differentiable and monotone on each of these intervals, we can use Proposition 3 to see that

$$\text{length } J_{n+1} = \int_{J_n} |F'_\mu(x)| dx \geq \int_{J_n} \lambda dx = \lambda \cdot \text{length } J_n.$$

It follows that the length of  $J_n$  grows exponentially. But this is impossible since  $J_n \subset [0, 1]$  for all  $n$ .

**3.** We will prove that  $\Lambda$  is perfect. It is enough to prove that for all  $x_0 \in \Lambda$  and all  $\epsilon > 0$  there is a  $y \in \Lambda$  so that  $x_0 \neq y$  and  $|x_0 - y| < \epsilon$ .

The set  $[0, 1] \setminus A_0$  consists of two closed intervals,

$$I_0 = \left[0, \frac{1}{2} - \frac{\sqrt{\mu-4}}{2\sqrt{\mu}}\right] \quad \text{and} \quad I_1 = \left[\frac{1}{2} + \frac{\sqrt{\mu-4}}{2\sqrt{\mu}}, 1\right].$$

The restrictions of  $F_\mu$  to  $I_0$  and  $I_1$ ,

$$F_\mu|_{I_0} : I_0 \rightarrow [0, 1] \quad \text{and} \quad F_\mu|_{I_1} : I_1 \rightarrow [0, 1],$$

are homeomorphisms, so they have inverse maps

$$h_0 : [0, 1] \rightarrow I_0 \quad \text{and} \quad h_1 : [0, 1] \rightarrow I_1.$$

Observe that if  $x \in \Lambda$  then  $h_0(x) \in \Lambda$  and  $h_1(x) \in \Lambda$ . This is because

$$x = F_\mu(h_0(x)) = F_\mu(h_1(x)).$$

So, the orbits of  $h_0(x)$  and  $h_1(x)$  follow the orbit of  $x$ .

By Proposition 3, we know that the restrictions  $F_\mu|_{I_0}$  and  $F_\mu|_{I_1}$  expand distance by a factor of  $\lambda > 1$ . So,  $h_0$  and  $h_1$  each contract distances by a factor of  $\lambda$ . That is, if  $a, b \in [0, 1]$ , then

$$|h_0(a) - h_0(b)| \leq \frac{1}{\lambda}|a - b| \quad \text{and} \quad |h_1(a) - h_1(b)| \leq \frac{1}{\lambda}|a - b|.$$

Choose any  $x_0 \in \Lambda$ . Then the orbit  $\{x_n = F_\mu^n(x_0) : n \geq 0\}$  is contained in  $\Lambda$  for all  $n \geq 0$ . Since  $\Lambda \subset I_0 \cup I_1$ , there is a sequence  $\langle s_n \in \{0, 1\} : n \geq 0 \rangle$  so that

$$F_\mu^n(x) \in I_{s_n} \quad \text{for all } n \geq 0.$$

In particular, notice that since  $F_\mu(x_n) = x_{n+1}$ , we know that  $x_n = h_{s_n}(x_{n+1})$ , since  $x_n \in I_{s_n}$  and the image of  $x_n$  under  $F_\mu$  is  $x_{n+1}$ . It then follows that

$$x_0 = h_{s_0} \circ \dots \circ h_{s_{n-2}} \circ h_{s_{n-1}}(x_n).$$

Now pick any  $\epsilon > 0$ . Then there is an  $n$  so that  $\frac{1}{\lambda^n} < \epsilon$ . Now choose  $y' = 0$  or  $y' = 1$ , so that  $y' \neq x_n$ . Note that both points lie in  $\Lambda$ , since they map to zero which is fixed by  $F_\mu$ . Define

$$y = h_{s_0} \circ \dots \circ h_{s_{n-2}} \circ h_{s_{n-1}}(y').$$

Since each map is a homeomorphism, we know that  $x_0 \neq y$ . Also these maps contract distances by  $\frac{1}{\lambda}$ . So

$$|x_0 - y| \leq \frac{1}{\lambda^n} |x_n - y'| \leq \frac{1}{\lambda^n} < \epsilon.$$

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THE CITY COLLEGE OF NEW YORK, NEW YORK, NY, USA 10031  
*E-mail address:* whooper@ccny.cuny.edu